Dependent Types and Category Theory

Because of Thanksgiving, there was no homework last week, so I'm putting a problem about dependent types on this week's homework.

1

Recall the definition of the type $\mathbb{N}$ of Natural Numbers:

$$n : \mathbb{N} := 0 \mid \mathtt{S}\ n$$

Then we can define **addition** of natural numbers as follows:

$$0 + n = n \mid (\mathtt{S}\ m) + n = \mathtt{S}\ (m + n)$$

Construct a value with the following type:[1]

$$\Pi_{n:\mathbb{N}}\ n + 0 = n$$

The following functions may be useful:

$$\begin{aligned} \mathtt{ind}_{\mathbb{N}} &: P(0) \to (\Pi_{m:\mathbb{N}}\ P(m) \to P(\mathtt{S}\ m)) \to \Pi_{n:\mathbb{N}}\ P(n) \\ \mathtt{transport} &: \Pi_{f:A\to B}\ \Pi_{x:A}\ \Pi_{y:A}\ (x = y) \to (fx = fy) \end{aligned}$$

2

In lecture we discussed the notion of a "Kleisli Category" where instead of our arrows being of the form $\alpha \to \beta$, we have some added (monadic) structure on our output type. Namely, our arrows have the form $\alpha \to \beta\ \mathtt{monad}$ for some fixed monad "$\alpha\ \mathtt{monad}$". Note that now arrow composition is not simple function composition anymore, instead it is a more complicated notion $\rightarrowtail$.

Prove that every monad "$\alpha\ \mathtt{monad}$" can be identified with a Kleisli Category by showing both that

$$\rightarrowtail : (\alpha \to \beta\ \mathtt{monad}) \to (\beta \to \gamma\ \mathtt{monad}) \to (\alpha \to \gamma\ \mathtt{monad})$$

is associative, namely $((f \rightarrowtail g) \rightarrowtail h) \cong (f \rightarrowtail (g \rightarrowtail h))$and that

$$\eta : \alpha \to \alpha\ \mathtt{monad}$$

acts as an identity arrow, namely $(f \rightarrowtail \eta) \cong (\eta \rightarrowtail f) \cong f$

Further, prove that every Kleisli Category can be identified with a monad by showing that given an associative function

$$\rightarrowtail : (\alpha \to \beta\ \mathtt{blah}) \to (\beta \to \gamma\ \mathtt{blah}) \to (\alpha \to \gamma\ \mathtt{blah})$$

and an identity under this function

$$\eta : \alpha \to \alpha\ \mathtt{blah}$$

then `blah` is a monad.

It may be useful to recall the definition of a monad:

An endofunctor `m` is called a **monad** iff it has two maps
$\eta : \alpha \to \alpha\ \mathtt{m}$
$\mu : \alpha\ \mathtt{m}\ \mathtt{m} \to \alpha\ \mathtt{m}$

So proving that `blah` is a monad amounts to showing it is an endofunctor
(namely there is some function $\mathtt{blah\_map} : (\alpha \to \beta) \to \alpha\ \mathtt{blah} \to \beta\ \mathtt{blah}$)
with two additional maps η and μ as defined above[2]

[1] Notice the definition of addition has 0 on the wrong side, so we cannot simply evaluate $n + 0$ to achieve the desired equality

[2] I'm glossing over some coherence conditions on η and μ, but I don't think you're losing anything by not explicitly seeing them