1

We want a value of type

$$\Pi_{n:\mathbb{N}}\ n + 0 = n$$

To construct such a term we will use induction. So if we can find terms

$$p_{\text{base}}\ :\ 0 + 0 = 0$$

and

$$p_{\text{inductive}}\ :\ \Pi_{m:\mathbb{N}}\ (m + 0 = m) \to (\mathtt{S}\ (m + 0) = \mathtt{S}\ m)$$

(representing the base case and inductive cases, respectively) then

$$\mathtt{ind}_{\mathbb{N}}\ p_{\text{base}}\ p_{\text{inductive}}\ :\ \Pi_{n:\mathbb{N}}\ n + 0 = n$$

Notice that by the definition of addition $0 + 0$ evaulates to 0, and so

$$\mathtt{refl}_0\ :\ 0 + 0 = 0$$

We will use this as our p_{base}.

Now we want a term which, for any m, takes a proof that $m + 0 = m$ and returns a proof that $\mathtt{S}\ m + 0 = \mathtt{S}\ m$. Notice, by the definition of addition

$$\mathtt{S}\ m + 0 = \mathtt{S}\ (m + 0)$$

Now since (by induction) we have a term $\alpha\ :\ m + 0 = m$, we want to "compose" α with $\mathtt{S}$ to get a term of type $\mathtt{S}\ (m + 0) = \mathtt{S}\ m$ (Recall from lecture that if two terms are equal, applying the same function to both terms preserves equality. This is exactly what the `transport` function tells us).

$$\mathtt{transport}\ \mathtt{S}\ (m + 0)\ m\ \alpha\ :\ \mathtt{S}\ (m + 0) = \mathtt{S}\ m$$

works.

Thus

$$\mathtt{ind}_{\mathbb{N}}\ \mathtt{refl}_0\ (\lambda m.\lambda\alpha.\mathtt{transport}\ \mathtt{S}\ (m + 0)\ m\ \alpha)\ :\ \Pi_{n:\mathbb{N}}\ n + 0 = n$$

As desired.

2

First we show Monad to Kleisli. Given

$$\begin{aligned} \mathtt{fmap}\ &:\ (\alpha \to \beta) \to M\alpha\ toM\beta \\ \mu\ &:\ (MM\alpha \to M\alpha) \\ \eta\ &:\ \alpha \to M\alpha \end{aligned}$$

we want to construct a term

$$\rightarrowtail\ :\ (\alpha \to M\beta) \to (\beta \to M\gamma) \to \alpha \to M\gamma$$

It is easy to verify

$$f \rightarrowtail g = \mu \circ \mathtt{fmap} g \circ f$$

has the desired type.

Now we show Kleisli to Monad. Given

$$\rightarrowtail\ :\ (\alpha \to M\beta) \to (\beta \to M\gamma) \to \alpha \to M\gamma$$

we want to construct

$$\mu\ :\ (MM\alpha \to M\alpha)$$

Again, it is easy to verify

$$\mu = id \rightarrowtail id$$

has the desired type.[1]

[1] Let the first id have type $MM\alpha \to MM\alpha$, and let the second id have type $M\alpha \to M\alpha$