98-317 Homework 13: Dependent Typing

Email solutions to cjwong@andrew.cmu.edu by the end of 2019-12-03.

Solve at least one of the following problems to the best of your ability. Incorrect or incomplete answers will be accepted if a reasonable attempt is present. *Please* do not spend more than 30 minutes on this homework if you don't want to. Post on Piazza if you need help or would like to discuss further.

In the following questions, we define $+ : \texttt{nat} \to \texttt{nat} \to \texttt{nat}$ to be

$$0 + m = n$$
$$S(n) + m = S(n + m)$$

where 0 and S are the same zero and successor functions that define the natural numbers.

Remember also that the value `Refl` has type $e_1 == e_2$ exactly when e_1 and e_2 are extensionally equivalent. However, you should not assume any properties of operations on natural numbers other than successor and induction.

1. Write down the type representing the proposition

$$\forall n \in \mathbb{N}. n + 0 == n$$

   You may assume that $\mathbb{N}$ and the type `nat` are equivalent.

2. Prove the above theorem in dependent type theory. That is, write down a term of the type you gave in part 1. If you use `Refl`, please explicitly write down its type.

   You may use without proof the following lemmas:

$$\texttt{transitivityOfEquality} : \prod_{a:\texttt{nat}} \prod_{b:\texttt{nat}} \prod_{c:\texttt{nat}} ((a == b) \to (b == c) \to (a == c))$$
$$\texttt{successorEquality} : \prod_{a:\texttt{nat}} \prod_{b:\texttt{nat}} ((a == b) \to (S(a) == S(b)))$$

3. [HIGH DIFFICULTY] Write down the type of the proposition that addition (as defined abofe) is commutative. Then, prove it by producing a term of that type. You may use the result and both lemmas from problem 2.