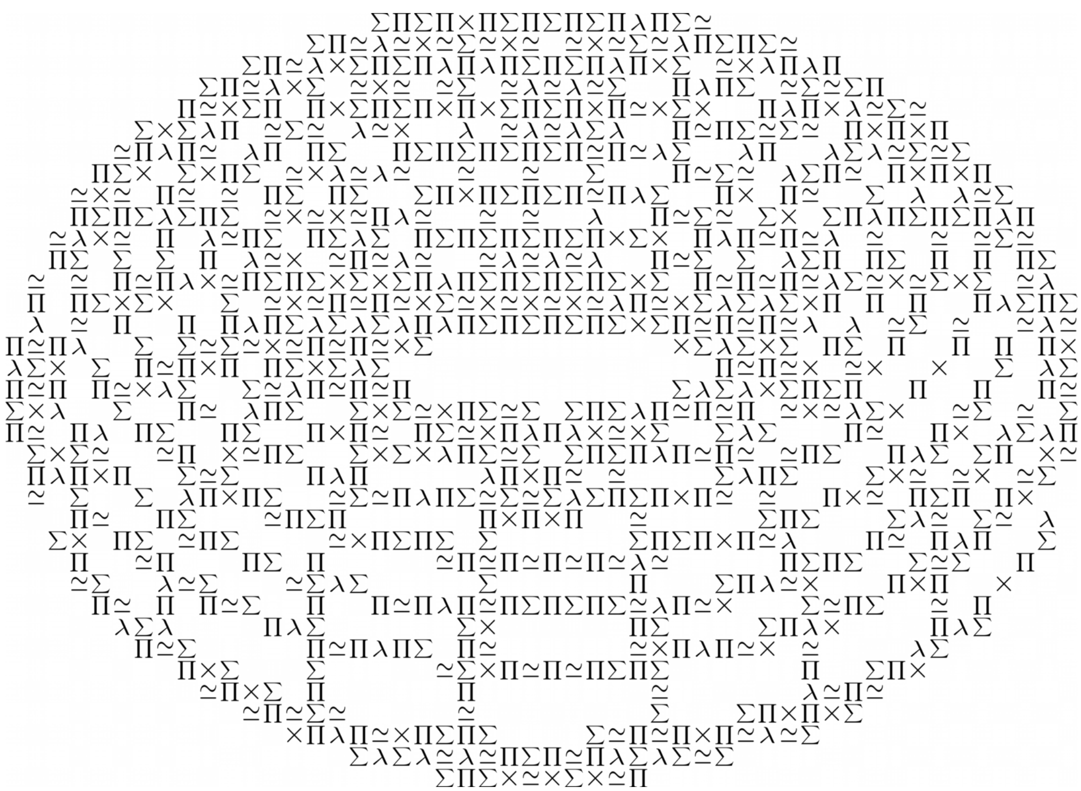
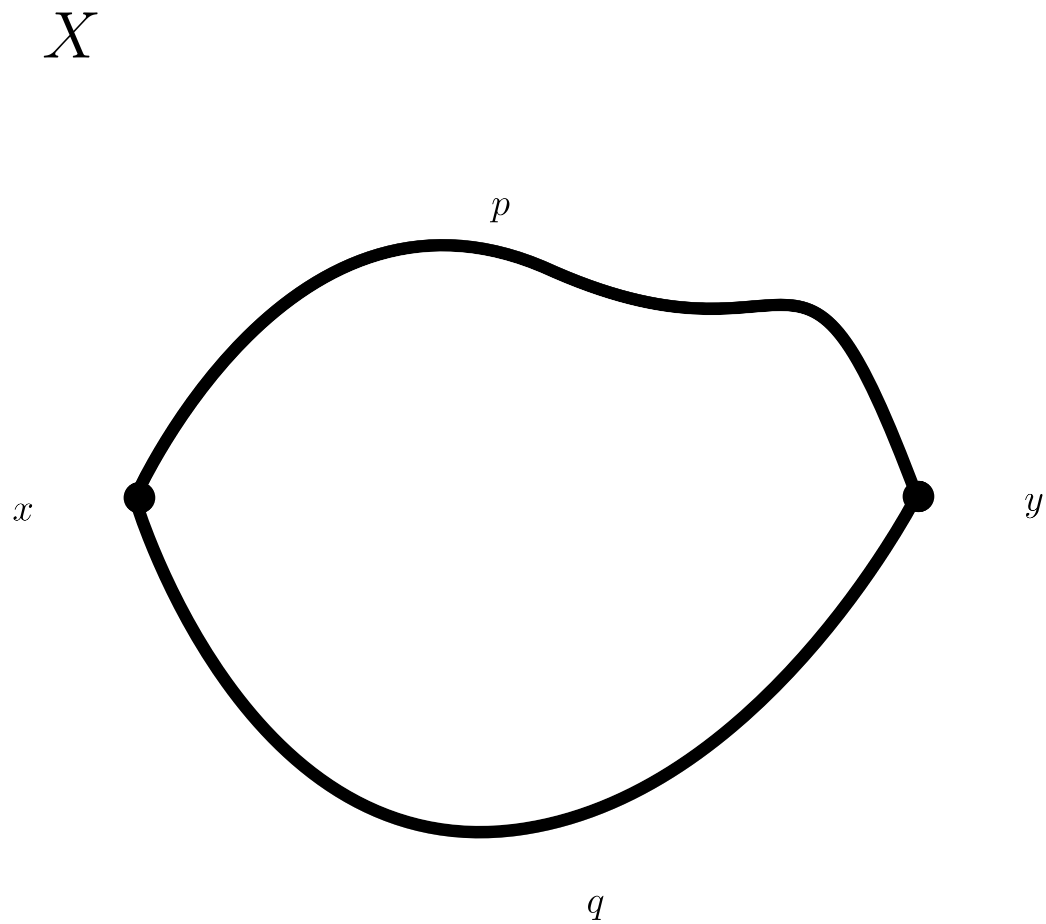
Homotopy Type Theory

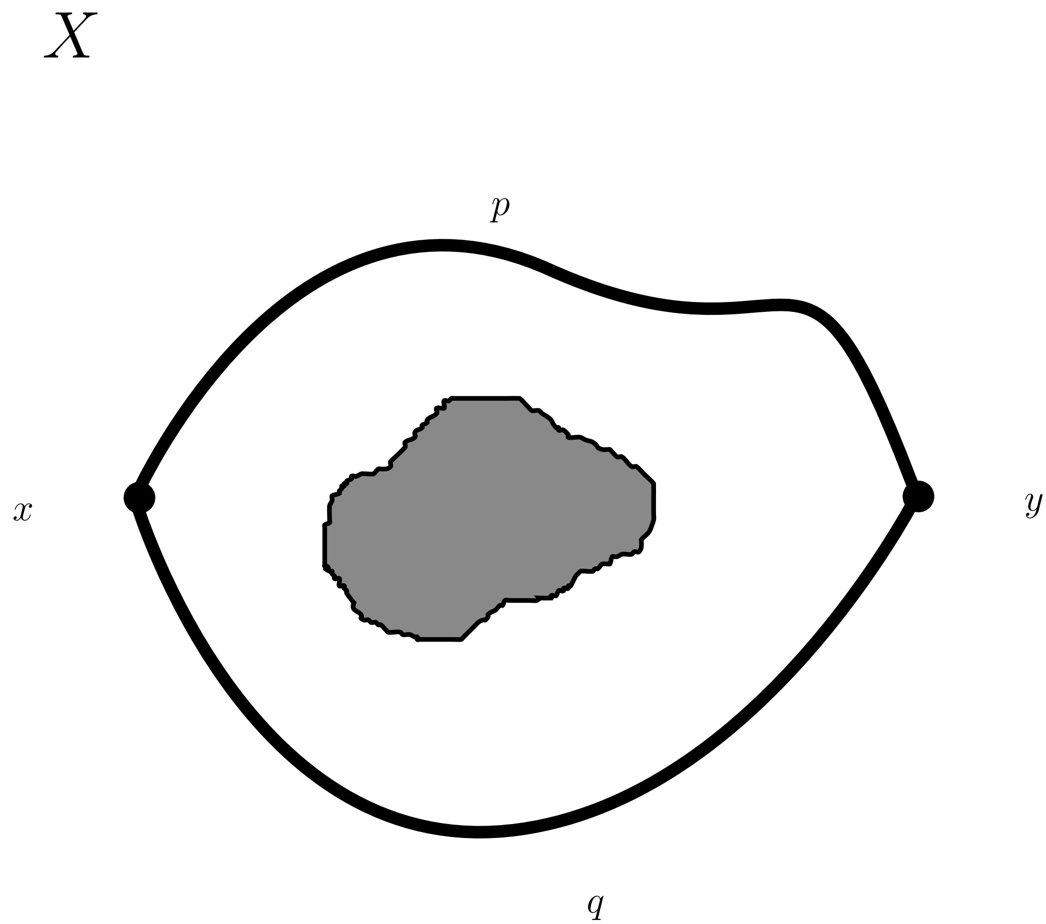
Hype For Types

03 December 2019



Based on Egbert Rijke's notes: https://hott.github.io/HoTT-2019/images/hott-intro-rijke.pdf


X
p
x
y
q


X
p
x
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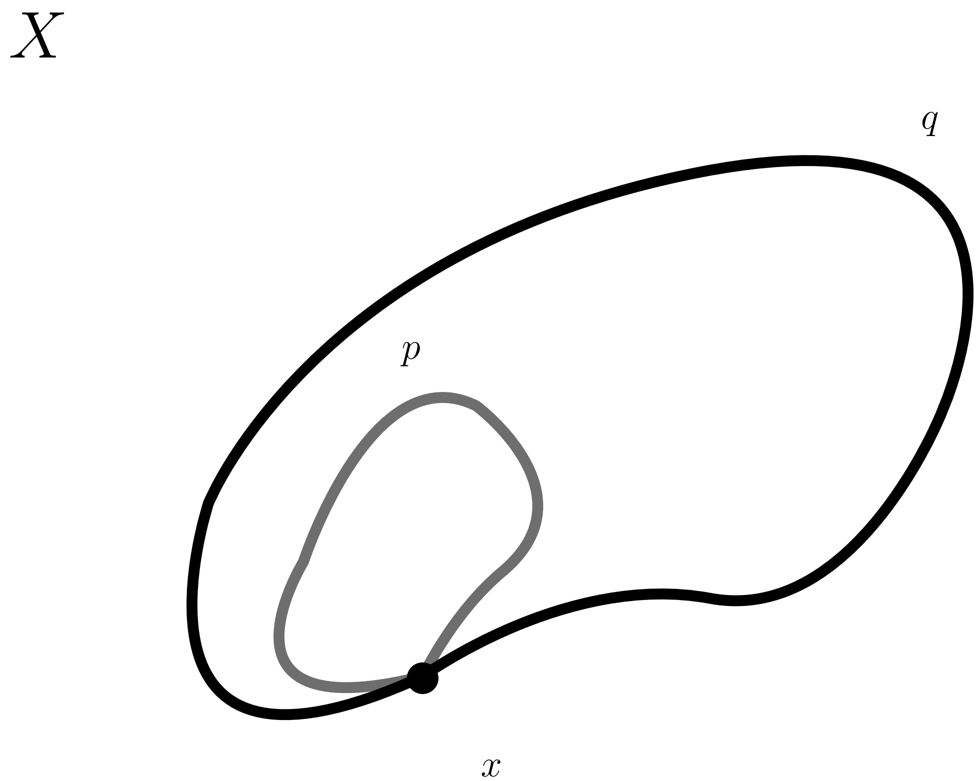
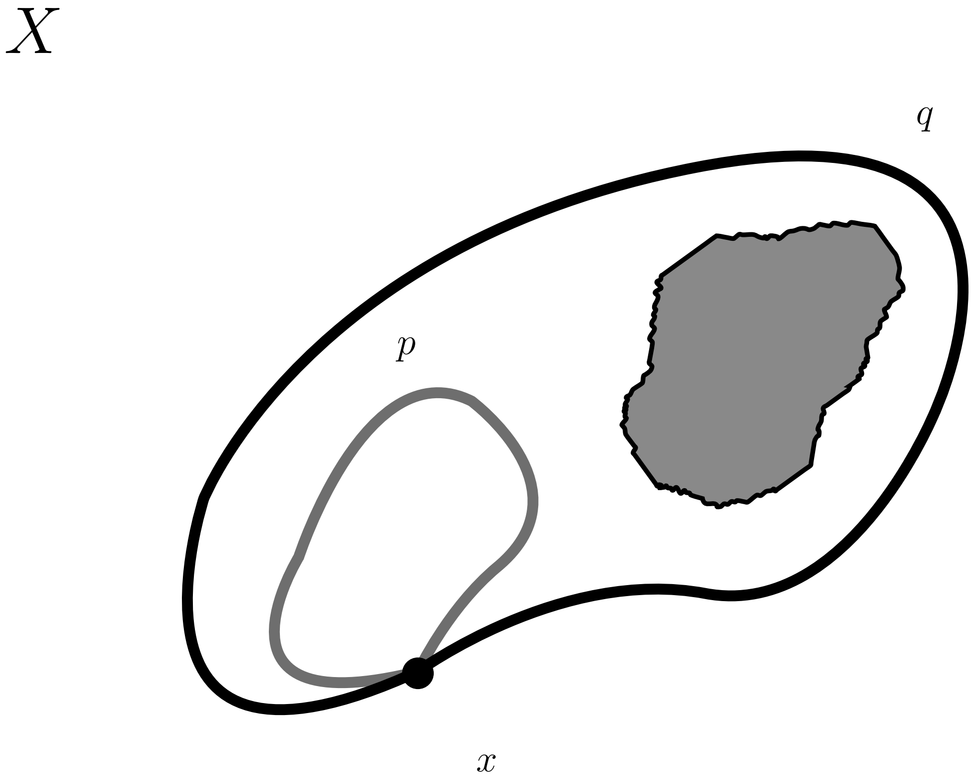
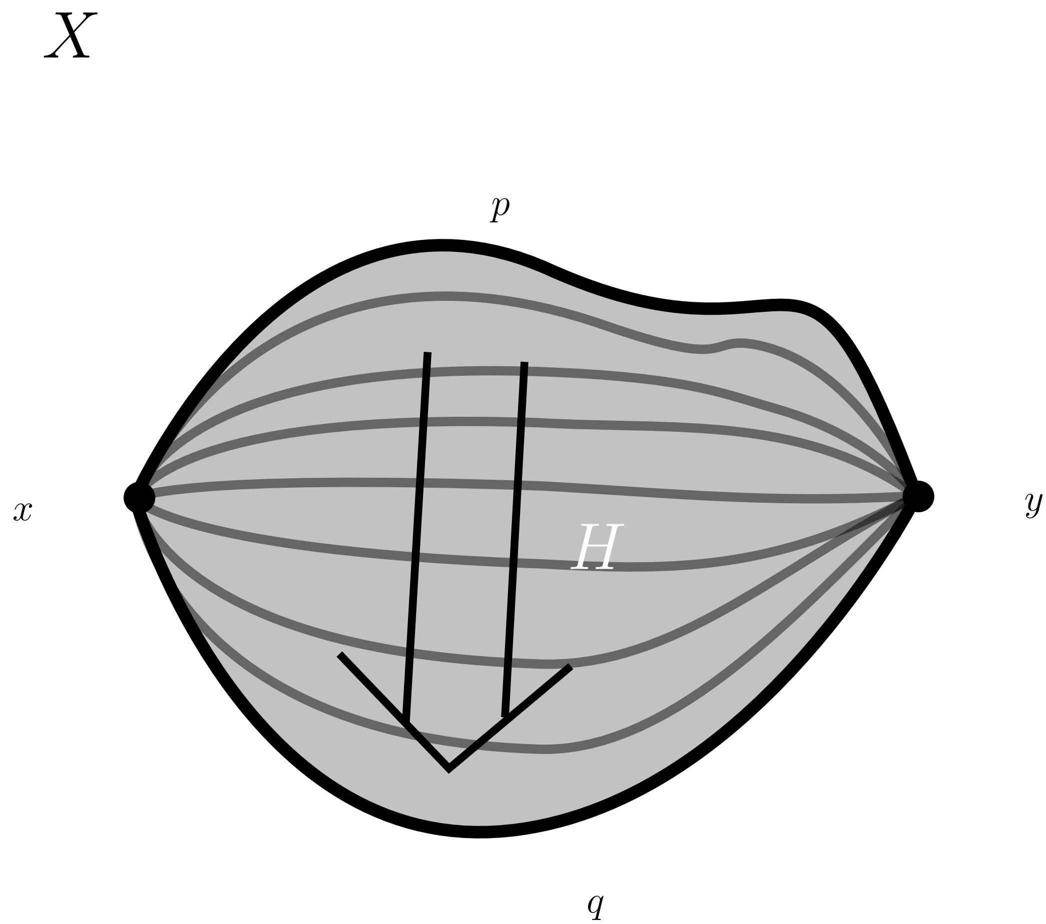
X
x
y



X
q
p
x



X
q
p
x



$$x \not\equiv y$$
$$p : x = y$$
$$q : x = y$$
$$p \not\equiv q$$
$$H : p = q$$

Recall: Contexts and judgements

$$\frac{\Gamma_1 \vdash \mathcal{J}_1 \qquad \Gamma_2 \vdash \mathcal{J}_2 \qquad \dots \qquad \Gamma_n \vdash \mathcal{J}_n}{\Gamma \vdash \mathcal{J}}$$

The Γs are finite lists of typing judgements $x_1 : A_1, \dots, x_k : A_k$ (a **context**), and the $\mathcal{J}$'s are of one of four forms:

- A type, as in:

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash B \ \mathsf{type}}{\Gamma \vdash A \to B \ \mathsf{type}}$$

- $x : \tau$, as in

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}$$

- $A \equiv B$, where A, B are types
- $a \equiv a' : A$, to mean a and a' are equivalent terms of type A, as in

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f \equiv \lambda x.f(x) : A \to B}$$

Example derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B} \qquad \dfrac{\dfrac{\Gamma \vdash B \ \mathsf{type}}{\Gamma, y : B \vdash \mathsf{id}_B(y) \equiv y : B}}{\Gamma, x : A, y : B \vdash \mathsf{id}_B(y) \equiv y : B}}{\dfrac{\Gamma, x : A \vdash \mathsf{id}_B(f(x)) \equiv f(x) : B}{\Gamma, x : A \vdash (\mathsf{id}_B \circ f)(x) \equiv f(x) : B}}}{\Gamma \vdash \mathsf{id}_B \circ f \equiv f : A \to B}$$

Type Families

B is called a "type family over A" whenever

$$\Gamma, x : A \vdash B \ \mathsf{type}$$

For some $t : A$, we write $B(t)$ for the type B given in context $\Gamma, t : A$.

The Fourfold Path:

Formation, Introduction, Elimination, Computation

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash B \ \mathsf{type}}{\Gamma \vdash A \to B \ \mathsf{type}}$$

$$\frac{\Gamma, x : A \vdash b(x) : B}{\Gamma \vdash \lambda x.b(x) : A \to B}$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}$$

$$\frac{\Gamma \vdash B \ \mathsf{type} \qquad \Gamma, x : A \vdash b(x) : B}{\Gamma, x : A \vdash (\lambda y.b(y))(x) \equiv b(x) : B} \qquad \frac{\Gamma \vdash f : A \to B}{\Gamma \vdash (\lambda x.f(x)) \equiv f : A \to B}$$

Dependent Product

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \prod_{x:A} B(x) \text{ type}}$$

$$\frac{\Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : \prod_{x:A} B(x)}$$

$$\frac{\Gamma \vdash f : \prod_{x:A} B(x)}{\Gamma, x : A \vdash f(x) : B(x)}$$

$$\frac{\Gamma, x : A \vdash B(x) \text{ type} \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma, x : A \vdash (\lambda y.b(y))(x) \equiv b(x) : B(x)} \qquad \frac{\Gamma \vdash f : \prod x : AB(x)}{\Gamma \vdash (\lambda x.f(x)) \equiv f : \prod_{x:A} B(x)}$$

Dependent Sum: Introduction and Elimination

$$\frac{\Gamma \vdash x : A \qquad \Gamma \vdash b : B(x)}{\Gamma \vdash (x, b) : \sum_{x:A} B(x)}$$

$$\frac{\Gamma \vdash t : \sum_{x:A} B(x)}{\Gamma \vdash p_1(t) : A} \qquad \frac{\Gamma \vdash t : \sum_{x:A} B(x)}{\Gamma \vdash p_2(t) : B(p_1(t))}$$

Other Useful Types

Nat

$$\frac{}{\vdash \mathbb{N}\ \mathsf{type}}$$

$$\frac{}{\vdash 0 : \mathbb{N}} \qquad \frac{}{\vdash \mathsf{succ} : \mathbb{N} \to \mathbb{N}}$$

$$\frac{\Gamma, n : \mathbb{N} \vdash P(n)\ \mathsf{type} \qquad \Gamma \vdash p_0 : P(0) \qquad \Gamma \vdash p_s : \prod_{n:\mathbb{N}} \big(P(n) \to P(\mathsf{succ}(n))\big)}{\Gamma \vdash \mathsf{ind}_{\mathbb{N}}(p_0, p_s) : \prod_{n:\mathbb{N}} P(n)}$$

$$\mathsf{ind}_{\mathbb{N}}(p_0, p_s)(0) \equiv p_0 \qquad \mathsf{ind}_{\mathbb{N}}(p_0, p_s)(\mathsf{succ}(n)) \equiv p_s(\mathsf{ind}_{\mathbb{N}}(p_0, p_s)(n))$$

For example,

$$\mathsf{fact} :\equiv \mathsf{ind}_{\mathbb{N}}(\mathsf{succ}(0), \lambda n.\lambda \mathsf{res}.n \cdot \mathsf{res}) : \prod_{n:\mathbb{N}} \mathbb{N}$$

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unit

$$\frac{}{\vdash \mathbf{1}\ \mathsf{type}}$$

$$\frac{}{\vdash \star : \mathbf{1}}$$

$$\mathsf{ind}_{\mathbf{1}} : P(\star) \to \prod_{x:\mathbf{1}} P(x)$$

$$\mathsf{ind}_{\mathbf{1}}(p)(\star) \equiv p$$

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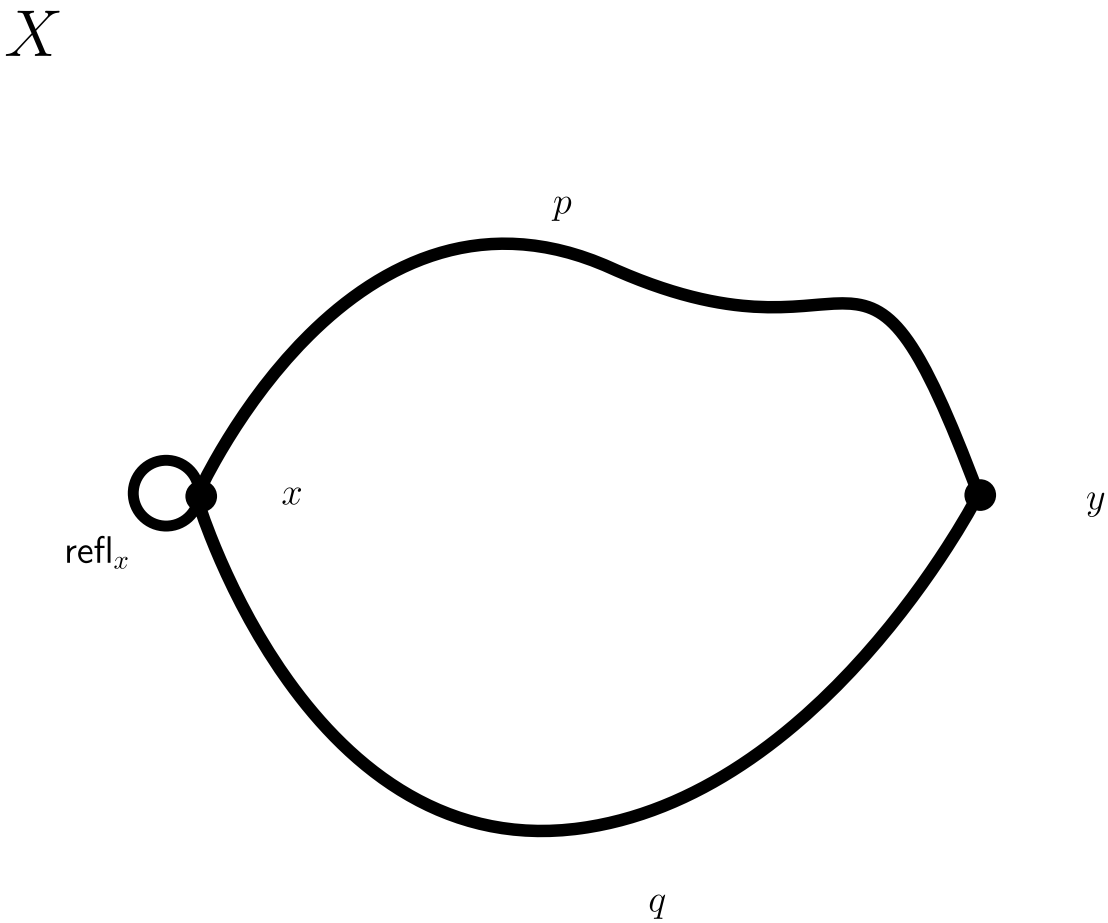
unit

$$\frac{}{\vdash \mathbf{0}\ \mathsf{type}}$$

$$\mathsf{ind}_{\mathbf{0}} : \prod_{x:\mathbf{0}} P(x)$$

Identity Types



$$\frac{\Gamma \vdash x : X}{\Gamma, y : X \vdash x = y \ \mathsf{type}}$$

$$\frac{\Gamma \vdash x : X}{\Gamma \vdash \mathsf{refl}_x : x = x}$$

Recall the induction principle for unit:

$$\mathsf{ind}_{\mathbf{1}} : P(\star) \to \prod_{x:\mathbf{1}} P(x)$$

So, letting $P(x)$ be $\star = x$, we can get:

$$\mathsf{ind}_{\mathbf{1}}(\mathsf{refl}_{\star}) : \prod_{x:\mathbf{1}} \star = x$$

and furthermore,

$$(\star, \mathsf{ind}_{\mathbf{1}}(\mathsf{refl}_{\star})) : \sum_{c:\mathbf{1}} \prod_{x:\mathbf{1}} c = x$$

A type X is called *contractible* if there's a term $\sum_{c:X} \prod_{x:X} c = x$

Recall the induction principle for void:

$$\mathsf{ind}_{\mathbf{0}} : \prod_{x:\mathbf{0}} P(x)$$

So, in particular, we can get a term of type

$$\prod_{x:\mathbf{0}} \prod_{y:\mathbf{0}} x = y$$

Moreover, this type is contractible. A type X is called a *proposition* if, for every $x, y : X$, $x = y$ is contractible.

Proposition 0.1

All contractible types are propositions, but not all propositions are contractible

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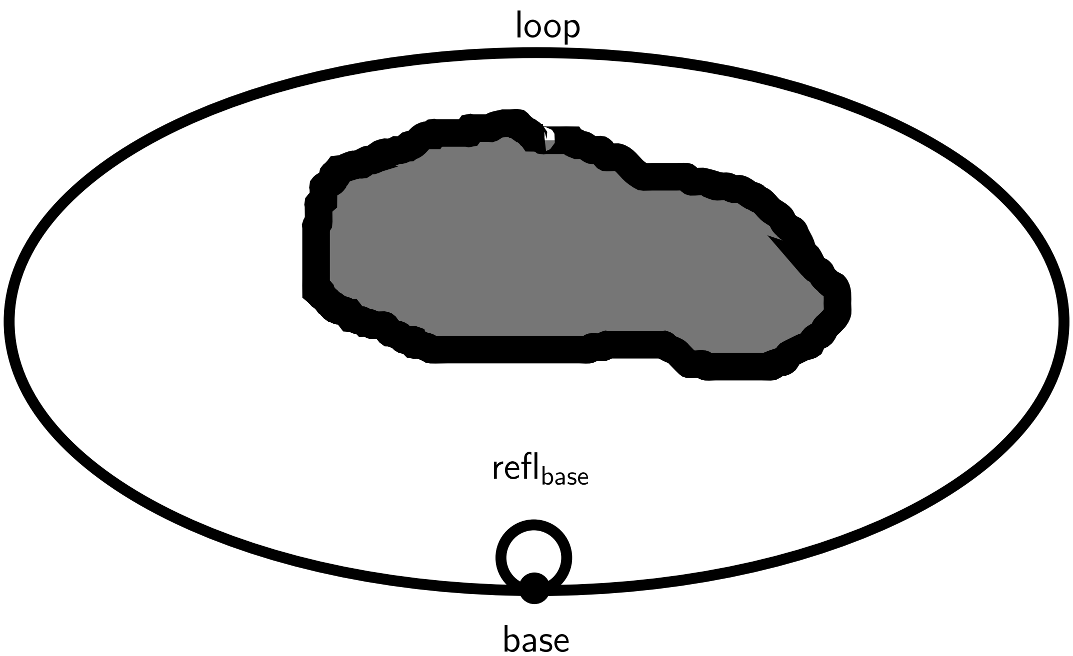
Proposition 0.2

For every $m, n : \mathbb{N}$, the identity type $m = n$ is a proposition.

If a type X is such that, for every $x, y : X$, the identity type $x = y$ is a proposition, then X is called a *set*.

Higher Inductive Types

$\mathbb{S}^1$



The type $\mathbb{S}^1$ is given by the constructors

$$\mathsf{base} : \mathbb{S}^1 \qquad \mathsf{loop} : \mathsf{base} = \mathsf{base}$$

Key Point: There is no term of type $\mathsf{loop} = \mathsf{refl}_{\mathsf{base}}$. So $\mathbb{S}^1$ is not a set.

Theorem 0.3

$$(\mathsf{base} = \mathsf{base}) \simeq \mathbb{Z}$$

Greatest HITs

- The interval $\mathbb{I}$ can be defined by

$$0 : \mathbb{I} \qquad 1 : \mathbb{I} \qquad \mathsf{inter} : 0 = 1$$

- The (2-)sphere – denoted $\mathbb{S}^2$ – can be defined by the constructors

$$\mathsf{N} : \mathbb{S}^2 \qquad \mathsf{S} : \mathbb{S}^2 \qquad \mathsf{merid} : \mathsf{N} = \mathsf{S} \qquad \mathsf{equator} : \mathsf{merid} = \mathsf{merid}$$

- The torus can be defined as

$$\mathbb{S}^1 \times \mathbb{S}^1$$