# Algebraic Data Types

Hype for Types

September 7, 2021

# Outline

- Look at types we already know from a different angle

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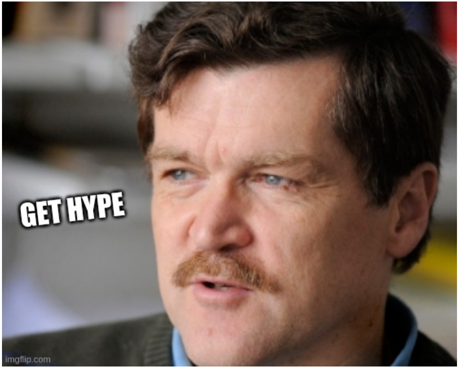

# Introduction to Counting

# bool and order

## Notation

Write $|\tau|$ to denote the number of elements in type $\tau$.

```
datatype bool  = false | true
datatype order = LESS | EQUAL | GREATER
```

What size are they?

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$$|\textbf{bool}| = 2$$
$$|\textbf{order}| = 3$$

Often, we refer to **bool** as 2 and **order** as 3:

$$\texttt{true} : 2$$
$$\texttt{LESS} : 3$$

# Products

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## Question

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For example,

$$\begin{aligned} |\textbf{bool} \times \textbf{order}| &= |\textbf{bool}| \times |\textbf{order}| \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

# What do you know!

**Theorem: Commutativity of Products**

For all $\tau_1, \tau_2$:

$$\tau_1 \times \tau_2 \simeq \tau_2 \times \tau_1$$

**Theorem: Associativity of Products**

For all $\tau_1, \tau_2, \tau_3$:

$$\tau_1 \times (\tau_2 \times \tau_3) \simeq (\tau_1 \times \tau_2) \times \tau_3$$

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*How* do we know?

# Proving Type Isomorphisms

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We write two (total) functions, $f : \tau \to \tau'$ and $f' : \tau' \to \tau$, such that $f$ and $f'$ are *inverses*.

$$\texttt{f' (f x)} \cong \texttt{x}$$

$$\texttt{f (f' x)} \cong \texttt{x}$$

# Associativity of Products: Proved!

Let's prove associativity of products:

$$\tau_1 \times (\tau_2 \times \tau_3) \simeq (\tau_1 \times \tau_2) \times \tau_3$$

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Need to write:

$$f : \tau_1 \times (\tau_2 \times \tau_3) \to (\tau_1 \times \tau_2) \times \tau_3$$
$$f' : (\tau_1 \times \tau_2) \times \tau_3 \to \tau_1 \times (\tau_2 \times \tau_3)$$

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Nice!

$$f = \texttt{fn (a,(b,c)) => ((a,b),c)}$$
$$f' = \texttt{fn ((a,b),c) => (a,(b,c))}$$

# Multiplicative Identity?

## Follow-Up

Is there an identity element, “1”?

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For all types $\tau$:

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| | |
|---|---|
| **SOME** $x$ | ($\tau$ choices) |
| **NONE** | (1 choice) |

# Sums

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datatype ('a,'b) either = Left of 'a | Right of 'b
```
[1]

[1] In the Standard ML Basis, (almost) the `Either` structure!

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$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \textbf{Left}\ e : \tau_1 + \tau_2}\ (\textsc{Left}) \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \textbf{Right}\ e : \tau_1 + \tau_2}\ (\textsc{Right})$$

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$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \qquad \Gamma, x_1 : \tau_1 \vdash e_1 : \tau \qquad \Gamma, x_2 : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \textbf{case}\ e\ \textbf{of}\ x_1 \Rightarrow e_1 \mid x_2 \Rightarrow e_2 : \tau}\ (\text{CASE})$$

**And of course...**

For all $\tau_1, \tau_2$:

$$|\tau_1 + \tau_2| = |\tau_1| + |\tau_2|$$

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# Options as Sums

```
datatype ('a,'b) either = Left of 'a | Right of 'b
```

Notice:

```
type 'a option = ('a,unit) either
```

We can represent $\tau$ **option** as $\tau +$ **unit**.

# Example: Distributivity

## Claim

For all types $A, B, C$:

$$(A \times B) + (A \times C) \simeq A \times (B + C)$$

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$f$ : `('a * 'b, 'a * 'c) either -> 'a * ('b,'c) either`

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$f$ = `fn Left (a,b) => (a,Left b) | Right (a,c) => (a,Right c)`

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**Practical Application**

Code refactoring principle! If both cases store the same data, factor it out.

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$$\frac{\Gamma \vdash e : \textbf{void}}{\Gamma \vdash \textbf{absurd}(e) : \tau}\ (\textsc{absurd})$$

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# Zero to Hero

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$$\frac{\Gamma \vdash e : \textbf{void}}{\Gamma \vdash \textbf{absurd}(e) : \tau}\ (\text{ABSURD})$$

## Implementing via SML Hacking

```
datatype void = Void of void
fun absurd (Void v) = absurd v
```

Notice: `absurd` is total!

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$f$ : `('tau,void) either -> 'tau`

$f'$ : `'tau -> ('tau,void) either`

$f$ = `fn Left x => x | Right v => absurd v`

$f'$ = `fn x => Left x`

= `Left`

# Functions

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**Theorem**

There are $|B|^{|A|}$ total functions from type $A$ to type $B$.

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Yes!

```
f  = Fn.uncurry : ('a -> 'b -> 'c) -> ('a * 'b -> 'c)
f' = Fn.curry   : ('a * 'b -> 'c) -> ('a -> 'b -> 'c)
```

# Recursive Types

# Lists

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```

$$L(\alpha) \simeq \textbf{unit} + \alpha \times L(\alpha)$$

$$\begin{aligned} L(\alpha) &= 1 + \alpha \times L(\alpha) \\ &= 1 + \alpha \times (1 + \alpha \times L(\alpha)) \\ &= 1 + \alpha + \alpha \times L(\alpha) \\ &= 1 + \alpha + \alpha \times (1 + \alpha \times L(\alpha)) \\ &= 1 + \alpha + \alpha^2 + \alpha^3 + \dots \end{aligned}$$

# Binary Trees

```
datatype 'a tree
  = Empty
  | Node of 'a tree * 'a * 'a tree
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$$
\begin{aligned}
T(\alpha) &\simeq \textbf{unit} + T(\alpha) \times \alpha \times T(\alpha) \\
&\simeq \textbf{unit} + \alpha \times T(\alpha)^2
\end{aligned}
$$

# Binary Shrubs

```
datatype 'a shrub
  = Leaf of 'a
  | Node of 'a shrub * 'a shrub
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Therefore, we would expect:

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```
f  = fn Zero => NONE | Succ n => SOME n
f' = fn NONE => Zero | SOME n => Succ n
```

haha type derivates go brrr

# Taking Things Too Far

**Question**

What is $\frac{d}{d\alpha}\tau(\alpha)$?

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## Smart Idea

Dismiss the idea outright - this is madness!

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## Our Plan

$>$:)

# >:)

$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$

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$$\alpha \times \alpha \times \alpha \qquad \mapsto \qquad 3 \times (\alpha \times \alpha)$$

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$$\alpha \times \alpha \times \alpha \qquad \mapsto \qquad 3 \times (\alpha \times \alpha)$$

**Conclusion**

Differentiating a power "eats" a tuple slot, and tells you which element was removed.

# Differentiating a List

Recall from calculus (?) that:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

[3] What the hype is a negative type?

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Recall from calculus (?) that:

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We have:[3]

$$L(\alpha) = 1 + \alpha + \alpha^2 + \ldots \stackrel{?}{=} \frac{1}{1-\alpha}$$

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# Differentiating a List

Recall from calculus (?) that:

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We have:[3]

$$L(\alpha) = 1 + \alpha + \alpha^2 + \ldots \stackrel{?}{=} \frac{1}{1-\alpha}$$

$$\begin{aligned} \frac{d}{d\alpha} L(\alpha) &= \frac{d}{d\alpha} \frac{1}{1-\alpha} \\ &= \frac{1}{(1-\alpha)^2} \\ &= \left( \frac{1}{1-\alpha} \right)^2 \\ &= L(\alpha)^2 \end{aligned}$$

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# Tree for Two, and Two for Tree

We said:

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Here we go again...

$$\begin{aligned}
\frac{d}{d\alpha} T(\alpha) &= \frac{d}{d\alpha} 1 + \frac{d}{d\alpha} \alpha T(\alpha)^2 \\
&= \alpha \times \frac{d}{d\alpha} T(\alpha)^2 + \frac{d}{d\alpha} \alpha \times T(\alpha)^2 \\
&= 2\alpha T(\alpha) \times \frac{d}{d\alpha} T(\alpha) + T(\alpha)^2 \\
\frac{d}{d\alpha} T(\alpha) &= T(\alpha)^2 \left( \frac{1}{1 - 2\alpha T(\alpha)} \right) \\
&= T(\alpha)^2 L(2\alpha T(\alpha))
\end{aligned}$$

# Holey Cow!

$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$
$$\frac{d}{d\alpha}L(\alpha) = L(\alpha)^2$$
$$\frac{d}{d\alpha}T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha))$$

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**Theorem**

The Derivative of a Regular Type is its Type of One-Hole Contexts.[a]

[a] `http://strictlypositive.org/diff.pdf`

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"punctured" tuple

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$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$ “punctured” tuple

$$\frac{d}{d\alpha}L(\alpha) = L(\alpha)^2$$ list zipper

$$\frac{d}{d\alpha}T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha))$$

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# Holey Cow!

$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2 \qquad \text{“punctured” tuple}$$

$$\frac{d}{d\alpha}L(\alpha) = L(\alpha)^2 \qquad \text{list zipper}$$

$$\frac{d}{d\alpha}T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha)) \qquad \text{tree zipper}$$

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- Generalized our type theory to include *sum types* (and **void**)
- Considered *recursive types*[4]
- Invented a type-level hole punch

[4] More on that later...