

Category Theory (for Programmers)

Hype for Types

October 12, 2021

What is a category?

Monoids

Definition

A *monoid* M is the data:

- type t

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- value $f : t \rightarrow t \rightarrow t$

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This abstraction is handy! e.g.:

```
Seq.reduce M.f M.z : t seq -> t
```

Examples of Monoids

There are many monoids. For example:

- Natural numbers with zero, addition
- Natural numbers with one, multiplication
- Strings with empty string, string concatenation
- Lists with empty list, appending
- Sets with empty set, union

Categories

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- a collection of objects, $\text{Ob}(\mathcal{C})$

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- for every object $x \in \text{Ob}(\mathcal{C})$, an arrow $\text{id}_x : x \rightarrow x$

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- for every object $x \in \text{Ob}(\mathcal{C})$, an arrow $\text{id}_x : x \rightarrow x$
- for every arrow $u : x \rightarrow y$ and $v : y \rightarrow z$, an arrow $u \circ v : x \rightarrow z$

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- for every arrow, a source $x \in \text{Ob}(\mathcal{C})$
- for every arrow, a target $y \in \text{Ob}(\mathcal{C})$
- for every object $x \in \text{Ob}(\mathcal{C})$, an arrow $\text{id}_x : x \rightarrow x$
- for every arrow $u : x \rightarrow y$ and $v : y \rightarrow z$, an arrow $u \circ v : x \rightarrow z$
- for every arrow $f : w \rightarrow x$, $g : x \rightarrow y$, $h : y \rightarrow z$,
 $f \circ (g \circ h) = (f \circ g) \circ h$

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There are many categories. For example:

- Objects are sets, arrows are functions
- Objects are groups, arrows are group homomorphisms
- Objects are “numbers”, arrows are for \leq
- Objects are propositions, arrows are implications
- Objects are SML types, arrows are (total) functions

We'll focus on the last one.

Mappables¹

¹Well, “functors”, but that’s already a thing in SML... 

From Category to Category

What would a transformation from category to category look like?

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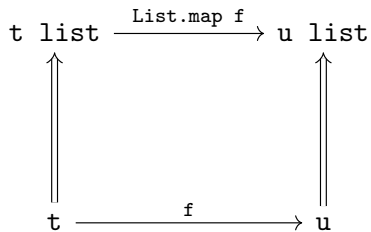
We must:

- turn objects into objects
- turn arrows into arrows

How about:

```
type 'a map_obj    = 'a list
fun      map_arr f = List.map f
```

Visualizing Lists



Mappables?

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- value `map : ('a -> 'b) -> 'a t -> 'b t`

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A *mappable* M is the data:

- type $'a \rightarrow t$
- value $\text{map} : ('a \rightarrow 'b) \rightarrow 'a \rightarrow t \rightarrow 'b \rightarrow t$

Mappables?

Definition?

A *mappable* M is the data:

- type `'a t`
- value `map : ('a -> 'b) -> 'a t -> 'b t`

In other words:

```
signature MAPPABLE =  
  sig  
    type 'a t  
    val map : ('a -> 'b) -> 'a t -> 'b t  
  end
```

Which map?

What if we picked:

```
type 'a map_obj    = 'a list

fun map_arr1 f =
  fn _ => []
fun map_arr2 f =
  fn l => List.map f (List.rev l)
fun map_arr3 f =
  fn []      => []
  | _::xs   => List.map f xs
```

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```

Problems:

```
map_arr Fn.id [1,2,3] =?= [1,2,3]
```

```
map_arr List.length o map_arr Int.toString
      =?=
```

```
map_arr (List.length o Int.toString)
```

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- upholds `map id = 'a t -> 'a t id`

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- upholds `map f o map g = map (f o g)`

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In other words:

```
signature MAPPABLE =  
  sig  
    type 'a t  
    val map : ('a -> 'b) -> 'a t -> 'b t  
    (* invariants: ... *)  
  end
```

Optimization: Loop Fusion!

If we have:

```
int [n] arr;  
  
for (int i = 0; i < n; i++)  
    arr[i] = f(i);  
  
for (int i = 0; i < n; i++)  
    arr[i] = g(i);
```

²Not just for lists - any data structure with a “sensible” notion of map works!


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for (int i = 0; i < n; i++)  
    arr[i] = g(i);
```

then it must be equivalent to:²

```
for (int i = 0; i < n; i++)  
    arr[i] = g(f(i));
```

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Some Example Mappables

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- Lists

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- Trees

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Some Example Mappables

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- Options
- Trees
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- Functions `int -> 'a`
- ...

i.e., (almost) anything polymorphic.

Conclusion

It's a useful abstraction.

Monads

Descent into partial madness

Partial functions return options

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- `sqrt : int -> int opt`

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- `tail : a list -> a list opt`

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Partial functions return options:

- `sqrt : int -> int opt`
- `div : (int * int) -> int opt`
- `head : a list -> a opt`
- `tail : a list -> a list opt`

How would we write the partial version of `tail_3`

```
(* tail_3 : a list -> a list *)  
fun tail_3 (_::_::_::L) = L
```

Composing partial functions

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tail_3 : 'a list -> 'a list opt
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tail_3 : 'a list -> 'a list opt
```

Partial madness!

```
fun tail_3 L0 =  
  case tail L0 of  
    NONE => NONE  
  | SOME L1 =>  
    ( case tail L1 of  
      NONE => NONE  
    | SOME L2 => tail L2)
```

Composing partial functions

How would we write the partial version of `tail_3`?

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tail_3 : 'a list -> 'a list opt
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Partial madness!

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```

What about `tail_5`?

Composing partial functions (again)

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tail_5 : 'a list -> 'a list opt
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If only...

```
val tail_5 = tail o tail o tail o tail o tail
```

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How would we write the partial version of `tail_5`?

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tail_5 : 'a list -> 'a list opt
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If only...

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val tail_5 = tail o tail o tail o tail o tail
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Another kind of compose

```
o : (b -> c) * (a -> b) -> a -> c
```

```
<=< : (b -> c opt) * (a -> b opt) -> a -> c opt
```

Composing partial functions (again)

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Another kind of compose

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o : (b -> c) * (a -> b) -> a -> c
```

```
<=< : (b -> c opt) * (a -> b opt) -> a -> c opt
```

Ta-da!

```
fun f <=< g =  
  (fn NONE => NONE | SOME x => f x) o g
```

More than a composition

Some useful versions of common tools

```
type 'a t = 'a option
```

Compose

```
val <=< : ('b -> 'c t) * ('a -> 'b t) -> ('a -> 'c t)
```

More than a composition

Some useful versions of common tools

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```

Compose

```
val <=< : ('b -> 'c t) * ('a -> 'b t) -> ('a -> 'c t)
```

Apply

```
val >>= : 'a t * ('a -> 'b t) -> 'b t
```

More than a composition

Some useful versions of common tools

```
type 'a t = 'a option
```

Compose

```
val <=< : ('b -> 'c t) * ('a -> 'b t) -> ('a -> 'c t)
```

Apply

```
val >>= : 'a t * ('a -> 'b t) -> 'b t
```

Flatten

```
val join : 'a t t -> 'a t
```

```
bind : 'a t * ('a -> 'b t) -> 'b t
```

```
type 'a t = 'a option
```

```
fun x >>= f = case x of SOME x => f x  
                    | NONE => NONE
```

```
bind : 'a t * ('a -> 'b t) -> 'b t
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fun x >>= f = case x of SOME x => f x  
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```
type 'a t = 'a list
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```
fun xs >>= f = List.concat (List.map f xs)
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```
type 'a t = 'a * string
```

```
fun (x,a) >>= f = let (y,b) = f x  
                  in (y,a^b) end
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bind : 'a t * ('a -> 'b t) -> 'b t
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```

```
type 'a t = 'a * string
```

```
fun (x,a) >>= f = let (y,b) = f x  
                  in (y,a^b) end
```

```
type 'a t = unit -> 'a
```

```
fun x >>= f = fn () => f (x()) ()
```

```
bind : 'a t * ('a -> 'b t) -> 'b t
```

```
type 'a t = 'a option  
fun x >>= f = case x of SOME x => f x  
                    | NONE => NONE
```

```
type 'a t = 'a list  
fun xs >>= f = List.concat (List.map f xs)
```

```
type 'a t = 'a * string  
fun (x,a) >>= f = let (y,b) = f x  
                    in (y,a^b) end
```

```
type 'a t = unit -> 'a  
fun x >>= f = fn () => f (x()) ()
```

```
datatype 'a t = Ret of 'a | Err of exn  
fun x >>= f = case x of Ret a => f x  
                    | Err x => Err x
```

Programming with Monads

```
readInput      : stream -> string option
parseUsername  : string -> string option
getUserFromId  : string -> user option
getAvatar      : user   -> image option
```

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```

SOME TextIO.stdin

```
>>= readInput
>>= parseUsername
>>= getUserFromId
>>= getAvatar
```

Parallel: Imperative Programming

```
inString <- SOME TextIO.stdIn
userId <- parseUsername inString
user <- getUserFromId userId
avatar <- getAvatar user
```

Useful pattern!

Key Idea

Monads are a useful programming tool!

```
signature MONAD =  
  sig  
    type 'a t  
    val return : 'a -> 'a t  
    val >>= : 'a t * ('a -> 'b t) -> 'b t  
  end
```