

Algebraic Data Types

Hype for Types

September 6, 2022

Outline

- Look at types we already know from a different angle

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- Formalize some important new type concepts

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- Formalize some important new type concepts
- break the universe

Introduction to Counting

bool and order

Notation

Write $|\tau|$ to denote the number of elements in type τ .

```
datatype bool = false | true
datatype order = LESS | EQUAL | GREATER
```

What size are they?

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$$|\mathbf{bool}| = 2$$

$$|\mathbf{order}| = 3$$

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What size are they?

$$|\mathbf{bool}| = 2$$

$$|\mathbf{order}| = 3$$

Often, we refer to **bool** as 2 and **order** as 3:

true : 2

LESS : 3

Products

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Question

What is $|\tau_1 \times \tau_2|$?

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$|\tau_1| \times |\tau_2|$ - hence, the notation.

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$|\tau_1| \times |\tau_2|$ - hence, the notation.

For example,

$$\begin{aligned} |\mathbf{bool} \times \mathbf{order}| &= |\mathbf{bool}| \times |\mathbf{order}| \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

What do you know!

Theorem: Commutativity of Products

For all τ_1, τ_2 :

$$\tau_1 \times \tau_2 \simeq \tau_2 \times \tau_1$$

Theorem: Associativity of Products

For all τ_1, τ_2, τ_3 :

$$\tau_1 \times (\tau_2 \times \tau_3) \simeq (\tau_1 \times \tau_2) \times \tau_3$$

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Proving Type Isomorphisms

To prove that $\tau \simeq \tau'$, we need a *bijection* between τ and τ' .

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We write two (total) functions, $f : \tau \rightarrow \tau'$ and $f' : \tau' \rightarrow \tau$, such that f and f' are *inverses*.

$$f' (f \ x) \cong x$$

$$f (f' \ x) \cong x$$

Associativity of Products: Proved!

Let's prove associativity of products:

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Need to write:

$$f : \tau_1 \times (\tau_2 \times \tau_3) \rightarrow (\tau_1 \times \tau_2) \times \tau_3$$

$$f' : (\tau_1 \times \tau_2) \times \tau_3 \rightarrow \tau_1 \times (\tau_2 \times \tau_3)$$

Associativity of Products: Proved!

Let's prove associativity of products:

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Need to write:

$$\begin{aligned} f &: \tau_1 \times (\tau_2 \times \tau_3) \rightarrow (\tau_1 \times \tau_2) \times \tau_3 \\ f' &: (\tau_1 \times \tau_2) \times \tau_3 \rightarrow \tau_1 \times (\tau_2 \times \tau_3) \end{aligned}$$

Nice!

$$\begin{aligned} f &= \text{fn } (a, (b, c)) \Rightarrow ((a, b), c) \\ f' &= \text{fn } ((a, b), c) \Rightarrow (a, (b, c)) \end{aligned}$$

Multiplicative Identity?

Follow-Up

Is there an identity element, “1”?

$$\tau \times 1 = \tau$$

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Theorem

For all types τ :

$$\tau \times \mathbf{unit} \simeq \tau$$

$$\mathbf{unit} \times \tau \simeq \tau$$

Sums

Increment

Question

Is there such thing as $\tau + 1$?

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Answer

Yes! τ **option**.

Increment

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Answer

Yes! τ **option**.

SOME x

(τ choices)

NONE

(1 choice)

Sums

`datatype ('a,'b) either = Left of 'a | Right of 'b`¹

¹In the Standard ML Basis, (almost) the `Either` structure!

Sums

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$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{Left} \ e : \tau_1 + \tau_2} \text{ (LEFT)}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{Right} \ e : \tau_1 + \tau_2} \text{ (RIGHT)}$$

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Sums

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$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \tau \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \mathbf{case} \ e \ \mathbf{of} \ x_1 \Rightarrow e_1 \mid x_2 \Rightarrow e_2 : \tau} \text{ (CASE)}$$

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And of course...

For all τ_1, τ_2 :

$$|\tau_1 + \tau_2| = |\tau_1| + |\tau_2|$$

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Options as Sums

```
datatype ('a,'b) either = Left of 'a | Right of 'b
```

Notice:

```
type 'a option = ('a,unit) either
```

We can represent τ **option** as $\tau + \mathbf{unit}$.

Example: Distributivity

Claim

For all types A, B, C :

$$(A \times B) + (A \times C) \simeq A \times (B + C)$$

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$f : ('a * 'b, 'a * 'c) \text{ either} \rightarrow 'a * ('b, 'c) \text{ either}$
 $f' : 'a * ('b, 'c) \text{ either} \rightarrow ('a * 'b, 'a * 'c) \text{ either}$

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$f = \text{fn Left } (a, b) \Rightarrow (a, \text{Left } b) \mid \text{Right } (a, c) \Rightarrow (a, \text{Right } c)$
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Practical Application

Code refactoring principle! If both cases store the same data, factor it out.

Zero to Hero

If we can add, what's 0?

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Implementing via SML Hacking

```
datatype void = Void of void
fun absurd (Void v) = absurd v
```

Notice: `absurd` is total!

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void*

Claim

For all types τ :

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$f : ('tau, void) \text{ either} \rightarrow 'tau$
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For all types τ :

$$\tau + \mathbf{void} \simeq \tau$$

$$f : ('tau, void) \text{ either} \rightarrow 'tau$$
$$f' : 'tau \rightarrow ('tau, void) \text{ either}$$
$$f = \text{fn Left } x \Rightarrow x \mid \text{Right } v \Rightarrow \text{absurd } v$$
$$f' = \text{fn } x \Rightarrow \text{Left } x$$
$$= \text{Left}$$

Functions

How Many Functions?

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Theorem

There are $|B|^{|A|}$ total functions from type A to type B .

Example: Power of a Power

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In terms of types, that would mean:

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Yes!

```
f = Fn.uncurry : ('a -> 'b -> 'c) -> ('a * 'b -> 'c)
```

```
f' = Fn.curry   : ('a * 'b -> 'c) -> ('a -> 'b -> 'c)
```

Recursive Types

Lists

```
datatype 'a list = Nil | Cons of 'a * 'a list
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type 'a list = (unit, 'a * 'a list) either
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$$L(\alpha) \simeq \mathbf{unit} + \alpha \times L(\alpha)$$

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type 'a list = (unit, 'a * 'a list) either
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$$L(\alpha) \simeq \mathbf{unit} + \alpha \times L(\alpha)$$

$$\begin{aligned}L(\alpha) &= 1 + \alpha \times L(\alpha) \\ &= 1 + \alpha \times (1 + \alpha \times L(\alpha)) \\ &= 1 + \alpha + \alpha \times L(\alpha) \\ &= 1 + \alpha + \alpha \times (1 + \alpha \times L(\alpha)) \\ &= 1 + \alpha + \alpha^2 + \alpha^3 + \dots\end{aligned}$$

Natural Numbers

How many natural numbers are there?

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datatype nat = Zero | Succ of nat
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datatype nat = Zero | Succ of nat
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$$\mathbf{nat} = \mathbf{unit} + \mathbf{nat}$$

$$\mathbf{nat} = 1 + 1 + 1 + \dots = \infty$$

Therefore, we would expect:

$$\infty = 1 + \infty$$

$$\mathbf{nat} \simeq \mathbf{nat\ option}$$

Natural Numbers

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datatype nat = Zero | Succ of nat
```

nat = unit + nat

nat = 1 + 1 + 1 + ... = ∞

Therefore, we would expect:

$\infty = 1 + \infty$

nat \simeq nat option

```
f = fn Zero => NONE | Succ n => SOME n
```

```
f' = fn NONE => Zero | SOME n => Succ n
```

Binary Trees

```
datatype 'a tree  
  = Empty  
  | Node of 'a tree * 'a * 'a tree
```

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$$\begin{aligned} T(\alpha) &\simeq \mathbf{unit} + T(\alpha) \times \alpha \times T(\alpha) \\ &\simeq \mathbf{unit} + \alpha \times T(\alpha)^2 \end{aligned}$$

Binary Shrubs

```
datatype 'a shrub  
  = Leaf of 'a  
  | Node of 'a shrub * 'a shrub
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Binary Shrubs

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$$\begin{aligned} S(\alpha) &\simeq \alpha + S(\alpha) \times S(\alpha) \\ &\simeq \alpha + S(\alpha)^2 \end{aligned}$$

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How many binary shrubs are there?

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Counting

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$$S(\alpha) = \frac{1 - \sqrt{1 - 4\alpha}}{2} \quad (\text{quadratic formula})$$

$$S(\alpha) = \alpha^1 + \alpha^2 + 2\alpha^3 + 5\alpha^4 + \dots + \frac{1}{n} \binom{2n-2}{n-1} \alpha^n + \dots$$

(Taylor series)

What does that even MEAN?

$$S(\alpha) = \alpha^1 + \alpha^2 + 2\alpha^3 + 5\alpha^4 + \dots + \frac{1}{n} \binom{2n-2}{n-1} \alpha^n + \dots$$

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Revelation

$\frac{1}{n} \binom{2n-2}{n-1}$ is the number of 'a' shrubs of n nodes!

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- This sequence is called the Catalan numbers

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Revelation

$\frac{1}{n} \binom{2n-2}{n-1}$ is the number of 'a shrubs of n nodes!

- This sequence is called the Catalan numbers
- This technique is called Generating Functions

haha type derivatives go brrr

Taking Things Too Far

Question

What is $\frac{d}{d\alpha}\tau(\alpha)$?

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Smart Idea

Dismiss the idea outright - this is madness!

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Our Plan

>:)

>:)

$$\frac{d}{d\alpha}\alpha^3 = \left(\frac{d}{d\alpha}\alpha \times \alpha \times \alpha\right) + \left(\alpha \times \frac{d}{d\alpha}\alpha \times \alpha\right) + \left(\alpha \times \alpha \times \frac{d}{d\alpha}\alpha\right)$$

>:)

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$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$

>:)

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$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$

$$\alpha \times \alpha \times \alpha \quad \mapsto \quad 3 \times (\alpha \times \alpha)$$

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$$\frac{d}{d\alpha}\alpha^3 = 3\alpha^2$$

$$\alpha \times \alpha \times \alpha \quad \mapsto \quad 3 \times (\alpha \times \alpha)$$

Conclusion

Differentiating a power “eats” a tuple slot, and tells you which element was removed.

Differentiating a List

Recall that:

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1 - r}$$

³What the hype is a negative type?

Differentiating a List

Recall that:

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

We have:³

$$L(\alpha) = 1 + \alpha + \alpha^2 + \dots \stackrel{?}{=} \frac{1}{1-\alpha}$$

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Differentiating a List

Recall that:

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

We have:³

$$L(\alpha) = 1 + \alpha + \alpha^2 + \dots \stackrel{?}{=} \frac{1}{1-\alpha}$$

$$\begin{aligned} \frac{d}{d\alpha} L(\alpha) &= \frac{d}{d\alpha} \frac{1}{1-\alpha} \\ &= \frac{1}{(1-\alpha)^2} \\ &= \left(\frac{1}{1-\alpha} \right)^2 \\ &= L(\alpha)^2 \end{aligned}$$

³What the hype is a negative type?

Tree for Two, and Two for Tree

We said:

$$T(\alpha) = 1 + \alpha T(\alpha)^2$$

Here we go again...

Tree for Two, and Two for Tree

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Here we go again...

$$\begin{aligned}\frac{d}{d\alpha} T(\alpha) &= \frac{d}{d\alpha} 1 + \frac{d}{d\alpha} \alpha T(\alpha)^2 \\ &= \alpha \times \frac{d}{d\alpha} T(\alpha)^2 + \frac{d}{d\alpha} \alpha \times T(\alpha)^2 \\ &= 2\alpha T(\alpha) \times \frac{d}{d\alpha} T(\alpha) + T(\alpha)^2 \\ \frac{d}{d\alpha} T(\alpha) &= T(\alpha)^2 \left(\frac{1}{1 - 2\alpha T(\alpha)} \right) \\ &= T(\alpha)^2 L(2\alpha T(\alpha))\end{aligned}$$

Holey Cow!

$$\frac{d}{d\alpha} \alpha^3 = 3\alpha^2$$

$$\frac{d}{d\alpha} L(\alpha) = L(\alpha)^2$$

$$\frac{d}{d\alpha} T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha))$$

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Theorem

The Derivative of a Regular Type is its Type of One-Hole Contexts.^a

^a<http://strictlypositive.org/diff.pdf>

Holey Cow!

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“punctured” tuple

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list zipper

$$\frac{d}{d\alpha} T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha))$$

Theorem

The Derivative of a Regular Type is its Type of One-Hole Contexts.^a

^a<http://strictlypositive.org/diff.pdf>

Holey Cow!

$$\frac{d}{d\alpha} \alpha^3 = 3\alpha^2$$

“punctured” tuple

$$\frac{d}{d\alpha} L(\alpha) = L(\alpha)^2$$

list zipper

$$\frac{d}{d\alpha} T(\alpha) = T(\alpha)^2 L(2\alpha T(\alpha))$$

tree zipper

Theorem

The Derivative of a Regular Type is its Type of One-Hole Contexts.^a

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- Used type equations and generating functions to count objects
- Invented a type-level hole punch

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