Dependent Types

Hype for Types

November 28, 2023

Safe Printing

Detypify

Consider these well typed expressions:

```
sprintf "nice"
sprintf "%d" 5
sprintf "%s,%d" "wow" 32
```

What is the type of `sprintf`?

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sprintf "%d" 5
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What is the type of `sprintf`? Well... it depends.

Types have types too

The type of `sprintf` *depends* on the value of the argument.
In order to compute the type of `sprintf`, we'll need to write a function that takes a string (List char), and returns a *type*!

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(* sprintf s : formatType s *)
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```
(* sprintf s : formatType s *)

fun formatType (s : char list) : type =
  case s of
    []                    => char list
  | ('%' :: 'd' :: cs) => (int -> formatType cs)
  | ('%' :: 's' :: cs) => (string -> formatType cs)
  | (_ :: cs)           => formatType cs
```

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(* formatType "%d and %s"  =  int -> string -> char list *)
(* sprintf "%d and %s" : int -> string -> char list *)
```

Quantification

Ok, we can express the type of `sprintf s` for some argument `s`, but what's the type of `sprintf`?

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What if we had universal quantification over *values*?

```
sprintf : (s : char list) -> formatType s
```

Curry-Howard Again

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2. $\forall x : t.A$
3. $\Pi_{x:\tau} A$

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Question:

Seems like we now have two arrow types:

1. Normal: $A \to B$.
2. Dependent: $(x : A) \to B$

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Do we need both? Nope!

$$A \to B \equiv (\_ : A) \to B$$

Some Rules

$$\frac{\Gamma, x : \tau \vdash e : A \quad \Gamma, x : \tau \vdash A : \mathit{Type}}{\Gamma \vdash \lambda(x : \tau)e : (x : \tau) \to A}$$

$$\frac{\Gamma \vdash e_1 : (x : \tau) \to A \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \ e_2 : [e_2/x]A}$$

Note on Notation

In SML we write type contructors on the *right*:

```
val cool : int list = [1,2,3,4]
```

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val cool : List Int = [1,2,3,4]
val a : A = (* omitted *)
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Question

What is the type of `List`?

```
List : Type -> Type
```

List is a function over types!

Types are values[1]

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Vectors Again

If we can write functions from values to types, can we define new type constructors which depend on *values*?

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```
inductive Vec : Type  Nat  Type
| nil  : (A : Type)  Vec A 0
| cons : (A : Type)  (n : Nat)
           A  Vec A n  Vec A (n+1)

def xs : Vec String 3 :=
  cons String 2 "hype" (
    cons String 1 (toString 4) (
      cons String 0 "types" (nil String)
    )
  )
```

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| nil  : (A : Type)  Vec A 0
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            A  Vec A n  Vec A (n+1)

def two := 1 + 0 + 1

def xs : Vec String (6 / two) :=
  cons String two "hype" (
    cons String 1 (toString 4) (
      cons String 0 "types" (nil String)
    )
  )
```

Vectors are actually usable now!

```
val append : (a : Type) -> (n m : Nat) ->
             Vec a n ->
             Vec a m ->
             Vec a (n + m)

val repeat : (a : Type) -> (n : Nat) ->
             a ->
             Vec a n

val filter : (a : Type) -> (n : Nat) ->
             (a -> bool) ->
             Vec a n ->
             Vec a ?? (* What should go here? *)
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```

Ponder

How do we describe the return value of `filter`?

Existential Crisis

For filter, we need to return the vector's length, *in addition* to the vector itself:

```
val filter : (a : Type) -> (n : Nat) ->
             (a -> bool) ->
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             Nat × Vec a ??
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             Nat × Vec a ??
```

We want to refer to the left value of a tuple, in the TYPE on the right.

Intuition: existential quantification!

There exists some `n : Nat`, such that we return `Vec a n`.

(We're constructivists, so exists means I actually give you the value)

Duality

$$(x : \tau) \times A \equiv \exists x : \tau . A$$

This type can also be written:

1. $\{x : \tau \mid A\}$
2. $\Sigma_{x:\tau} A$

As before, $A \times B \equiv (\_ : A) \times B$

```
val filter : (a : Type) -> (n : Nat) ->
             (a -> bool) ->
             Vec a n ->
             (m : Nat) × Vec a m
```

More Rules

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : [e_1/x]A \quad \Gamma, x : \tau \vdash A : \mathit{Type}}{\Gamma \vdash (e_1, e_2) : (x : \tau) \times A}$$

$$\frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_1\ e : \tau} \qquad \frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_2\ e : [\pi_1\ e/x]A}$$

Ok, so what?

Specifications are actually pretty nice

Discussion

Do you actually read function contracts/specifications in 122/150?

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```
(* REQUIRES : input sequence is sorted *)
val search : int -> int seq -> int option

> search 3 [5,4,3] ==> NONE
(* "search is broken!" *)
(* piazza post ensues *)
```

Compile-time Contracts

The 122 solution:

```
int search (int target, int[] arr)
//@requires is_sorted(arr)
{
    ...
}
```

Nice, but only works at runtime.

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    ...
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```

Nice, but only works at runtime.

What if passing `search` a non-sorted list was a *type error*?

A simpler example

```
(* REQUIRES : second argument is greater than zero *)
val div : Nat -> Nat -> Nat
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val div : Nat -> (n : Nat) × (1 ≤ n) -> Nat
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Dividing by zero is impossible! And we incur no runtime cost to prevent it.

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```
val div : Nat -> (n : Nat) × (1 ≤ n) -> Nat
```

Dividing by zero is impossible! And we incur no runtime cost to prevent it.
What does a value of type $(n : Nat) \times (1 \leq n)$ look like?

$$(3, \text{conceptsHW1.pdf}) : (n : Nat) \times (1 \leq n)$$

Question:

What goes in the PDF?

15-151 Refresher

What constitutes a proof of $n \leq m$?

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We just have to define what $(\leq)$ means!

1. $\forall n,\ n \leq n$
2. $\forall m\ n,\ n \leq m \Rightarrow n + 1 \leq m + 1$

This looks familiar!

15-151 Refresher

What constitutes a proof of $n \leq m$?

We just have to define what $(\leq)$ means!

1. $\forall n,\ n \leq n$
2. $\forall m\ n,\ n \leq m \Rightarrow n + 1 \leq m + 1$

This looks familiar!

```
inductive Le : Nat  Nat  Prop
| refl {n : Nat} : Le n n
| step {n m : Nat} : Le n m  Le n (Nat.succ m)
```

conceptsHW1.pdf

```
inductive Le : Nat  Nat  Prop
| refl {n : Nat} : Le n n
| step {n m : Nat} : Le n m  Le n (Nat.succ m)

def ex1  : Le 0 0 := @Le.refl 0
def ex1' : Le 0 0 := Le.refl

def ex2 : Le 0 3 :=
  Le.step (Le.step (Le.step Le.refl))

def ex3 : Le 1 3 := Le.step (Le.step Le.refl)

def ex4 : (n : Nat) »' (Le 1 n) :=
  3, Le.step (Le.step Le.refl)
```

Red-black Trees

A kind of balanced binary tree of the following invariants:

- Every node is either red or black;
- Every red node must have two black children;
- Every leaf is black;
- The number of black nodes from the root to every leaf is the same.

Red-black Trees

The best you can do in SML is:

```
datatype Color = Red | Black

datatype 'a Tree =
          Empty
        | Node of Color * 'a * 'a Tree * 'a Tree
```

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          Empty
        | Node of Color * 'a * 'a Tree * 'a Tree
```

But there is nothing that stop me from building a bad tree:

```
Node (Red, 1, Node (Red, 2, Leaf, Leaf), Empty)
```

Dependent Type to Rescue: Red-black Trees

```
inductive Color
| black
| red

inductive RBT : Type  Color  Nat  Type
| leaf  : (A : Type)  RBT A black 0
| red : (A : Type)  (n : Nat)
       RBT A black n  A  RBT A black n  RBT A red n
| black : (A : Type)  (n : Nat)  (y1 y2 : Color)
       RBT A y1 n  A  RBT A y2 n  RBT A black (n+1)
```

Some Sort of Contract

```
inductive Sorted : List Nat  Prop
| nil_sorted    : Sorted []
| single_sorted : (n : Nat)  Sorted [x]
| cons_sorted   : (n m : Nat)
                    (xs : List Nat)
                    Le n m
                    Sorted (m :: xs)
                    Sorted (n :: m :: xs)

def search : Nat
             (xs : List Nat)
             Sorted xs
             Option Nat := sorry
```

A Type for Term Equality

If we can express a relation like $\leq$ and sortedness, how about equality?

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```
inductive Eq (A : Type) : A   A   Prop
| refl (a : A) : Eq A a a

def symm (A : Type) (x y : A) : Eq A x y   Eq A y x
  | Eq.refl x => Eq.refl

def trans (A : Type) (x y z : A)
          (h1 : Eq A x y) (h2 : Eq A y z)
          : Eq A x z :=
  match h1 with
  | Eq.refl x => h2

def plus_comm : (n m : Nat)   Eq Nat (n + m) (m + n) := sorry

def inf_primes : (n : nat)
               (m : Nat) »' ((m > n) » (Prime m)) := sorry
```