Constructive Logic

*-317

September 16, 2024

Proofs

Existence

I want to prove there exists a set with property P.
Is one of these more useful?

- Proof by contradiction: If such a set did not exist, we'd have a contradiction (insert proof here), therefore it must exist
- Direct proof: The set S has property P (insert proof here)

Existence

I want to prove there exists an algorithm to convert SML into x86 assembly.

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Existence

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- Proof by contradiction: If such a compiler did not exist, we'd have a contradiction (insert proof here), therefore it must exist
- Direct proof : CakeML (formally verified SML compiler)

To Construct or Not To Construct

Two kinds of proofs

- *Non-Constructive* : demonstrate the existence of a mathematical object, but *without* telling you what it is
- *Constructive* : demonstrate the existence of a mathematical object precisely by presenting an object and proving it has the desired properties

A Rational Example

Does there exist $a, b : \mathbb{R}$ such that a, b irrational but a^b rational?

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- *Non-Constructive*
  If $\sqrt{2}^{\sqrt{2}}$ rational, we're done. Otherwise $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$.

A Rational Example

Does there exist $a, b : \mathbb{R}$ such that a, b irrational but a^b rational?

- *Non-Constructive*
  If $\sqrt{2}^{\sqrt{2}}$ rational, we're done. Otherwise $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$.
- *Constructive*
  Take $a = \sqrt{2}$ and $b = log_2 9$. Then $\sqrt{2}^{log_2 9} = 9^{log_2 \sqrt{2}} = 9^{\frac{1}{2}} = 3$

Non-constructive proofs are frustrating in the real world

- Brouwer's Fixed Point Theorem
- Expander Graphs
- Ramsey Theory

Constructive proofs are useful to computer scientists

Constructive proofs provide *algorithms*!

A proof that all natural numbers have property P must describe a way to construct a proof of $P(n)$ for each $n : \mathbb{N}$

Formalization B)

- “You have to construct something” is pretty vague
- How do we formalize what it means for a proof to be constructive?

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- "You have to construct something" is pretty vague
- How do we formalize what it means for a proof to be constructive?
  1. Decide what kinds of proposition we want to talk about
  2. Inference rules!

Formalization B)

A few reasonable kinds of proposition

- $\top$
- $\bot$
- $A \wedge B$
- $A \vee B$
- $A \supset B$
- $\neg A$

Constructive Logic: Inference Rules

Conjunction ($\wedge$)

To get $A \wedge B$ (*introduction*), we need...

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$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \; (\wedge I)$$

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Given $A \wedge B$, we can extract two facts (*elimination*)... A and B:

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

Implication ($\supset$)

To get $A \supset B$, we need...

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To get $A \supset B$, we need... a proof of B *assuming* a proof of A:

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Implication ($\supset$)

To get $A \supset B$, we need... a proof of B *assuming* a proof of A:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \ (\supset I)$$

Given $A \supset B$ and A, we can extract... B:

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \ (\supset E)$$

Disjunction ($\vee$)

To get $A \vee B$, we need...

Disjunction ($\vee$)

To get $A \vee B$, we need... a proof of A *or* a proof of B:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee I_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee I_2)$$

Disjunction ($\vee$)

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Given $A \vee B$, we can extract... nothing?

Disjunction ($\vee$)

To get $A \vee B$, we need... a proof of A *or* a proof of B:

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Given $A \vee B$, we can extract... nothing? But if we *also* have "given A, then C" and "given B, then C," we can get C:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \; (\vee E)$$

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need...

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need... nothing!

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

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To get $\top$, we need... nothing!

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Can we get any information out of $\top$?

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need... nothing!

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

Can we get any information out of $\top$? No!

How can we get $\bot$?

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need... nothing!

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

Can we get any information out of $\top$? No!

How can we get $\bot$? We can't!

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need... nothing!

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

Can we get any information out of $\top$? No!

How can we get $\bot$? We can't!
But given $\bot$, we can obtain a proof of...

Truth ($\top$) and Falsehood ($\bot$)

To get $\top$, we need... nothing!

$$\frac{}{\Gamma \vdash \top}\ (\top I)$$

Can we get any information out of $\top$? No!

How can we get $\bot$? We can't!
But given $\bot$, we can obtain a proof of... anything!

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot E)$$

What about Negation ($\neg$)?

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What counts as a proof of $\neg A$?
We need to show something like "it's impossible to prove A"
Do we need new inference rules?
No!

$$\neg A \equiv A \supset \perp$$

$\neg A$ means "If we can prove A, we can do something impossible".

All the rules!

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge I) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ (\supset I) \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\supset E) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee I_1)$$

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Question

Does this seem familiar...?

Programs are Proofs

Back to the Simply-Typed Lambda Calculus

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Back to the Simply-Typed Lambda Calculus

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Let's Prove Some Stuff!

Some Examples

Theorem: Identity

Prove $A \supset A$.

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$\lambda f : A \times B \rightarrow C.\ \lambda a : A.\ \lambda b : B.\ f\ \langle a, b \rangle$, or `Fn.curry` in SML

More Examples

Theorem

Prove $\bot \vee \top$.

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Right $\langle\rangle$

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Theorem: Distributivity

Prove $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$.

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$\lambda x : A \times (B + C).$ **case snd**(x) **of** $x_1 \Rightarrow$ **Left** $\langle$**fst**$(x), x_1\rangle \mid x_2 \Rightarrow$
Right $\langle$**fst**$(x), x_2\rangle$
$\lambda x : (A \times B) + (A \times C).$ **case** x **of** $x_1 \Rightarrow \langle$**fst**(x_1), **Left snd**$(x_1)\rangle \mid x_2 \Rightarrow$
$\langle$**fst**(x_2), **Right snd**$(x_2)\rangle$

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Prove $A \wedge \neg A \supset B$. In other words, $A \wedge (A \supset \bot) \supset B$.

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Is there anything we can't prove constructively?

- Law of Excluded Middle : $P \vee \neg P$
- Double Negation Elimination : $\neg\neg P \supset P$

(These are actually equivalent)

So what?

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Question

These proofs seem pretty boring, can a type system express more complicated propositions?