Category Theory

1

Write three (distinct) functions $\varphi_1, \varphi_2, \varphi_3$ of type α `option` $\to \alpha$ `list`

2

Prove that each φ_i from above satisfies the naturality criterion, namely:

$$(\forall f : \alpha \to \beta)\,(\texttt{o\_map}\ f \circ \varphi_i \cong \varphi_i \circ \texttt{map}\ f)$$

Where, as usual,

$\texttt{o\_map} : (\alpha \to \beta) \to \alpha\ \texttt{option} \to \beta\ \texttt{option}$
$\texttt{o\_map f NONE} = \texttt{NONE}$
$\texttt{o\_map f (SOME x)} = \texttt{SOME (f x)}$

$\texttt{map} : (\alpha \to \beta) \to \alpha\ \texttt{list} \to \beta\ \texttt{list}$
$\texttt{map f [ ]} = \texttt{[ ]}$
$\texttt{map f (x :: xs)} = \texttt{(fx) :: (map f xs)}$

3

In lecture we discussed the notion of a "Kleisli Category" where instead of our arrows being of the form $\alpha \to \beta$, we have some added (monadic) structure on our output type. Namely, our arrows have the form $\alpha \to \beta$ `monad` for some fixed monad "α `monad`". Note that now arrow composition is not simple function composition anymore, instead it is a more complicated notion $\rightarrowtail$.

Prove that every monad "α `monad`" can be identified with a Kleisli Category by showing both that

$$\rightarrowtail : (\alpha \to \beta\ \texttt{monad}) \to (\beta \to \gamma\ \texttt{monad}) \to (\alpha \to \gamma\ \texttt{monad})$$

is associative, namely $((f \rightarrowtail g) \rightarrowtail h) \cong (f \rightarrowtail (g \rightarrowtail h))$and that

$$\eta : \alpha \to \alpha\ \texttt{monad}$$

acts as an identity arrow, namely $(f \rightarrowtail \eta) \cong (\eta \rightarrowtail f) \cong f$

Further, prove that every Kleisli Category can be identified with a monad by showing that given an associative function

$$\rightarrowtail : (\alpha \to \beta\ \texttt{blah}) \to (\beta \to \gamma\ \texttt{blah}) \to (\alpha \to \gamma\ \texttt{blah})$$

and an identity under this function

$$\eta : \alpha \to \alpha\ \texttt{blah}$$

then `blah` is a monad.

It may be useful to recall the definition of a monad:

An endofunctor `m` is called a **monad** iff it has two maps
$\eta : \alpha \to \alpha\ \texttt{m}$
$\mu : \alpha\ \texttt{m m} \to \alpha\ \texttt{m}$

So proving that `blah` is a monad amounts to showing it is an endofunctor
(namely there is some function $\texttt{blah\_map} : (\alpha \to \beta) \to \alpha\ \texttt{blah} \to \beta\ \texttt{blah}$)
with two additional maps η and μ as defined above[1]

[1] I'm glossing over some coherence conditions on η and μ, but I don't think you're losing anything by not explicitly seeing them