Seven Trees in One

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98-317 Hype For Types

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```
datatype tree = E | N of tree * tree

val foo : (tree * tree * tree * tree * tree * tree * tree -> tree) =
  fn (a as N _,b,c,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b as N _,c,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b,c as N _,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b,c,d as N _,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (E,E,E,E,N(e1,e2),f,g) => N(N(N(N(E,g),f),e1),e2)
   | (E,E,E,E,E,f as N _,g) => N(N(N(N(N(f,g),E),E),E),E)
   | (E,E,E,E,E,E,N(N(N(N(g1,g2),g3),g4),g5)) => N(N(N(N(N(E,g1),g2),g3),g4),g5)
   | (E,E,E,E,E,E,g) => g
```

Notice anything?

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What's the length of the left spine in the output?

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- $\geq$6 in the first four cases
- 4 in the fifth case
- $\geq$6 in the sixth case (`f` is non-empty)
- 5 in the seventh case
- $\leq$3 in the last case

Notice anything?

Another note:
In the first four cases, at least one node on the left spine must have a right child, but in the sixth case, that isn't true.

Notice anything?

- `foo` is injective!

Notice anything?

Also, `foo` is surjective.

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Also, `foo` is surjective. (prove it for yourself, very long and not very interesting)

Notice anything?

- `foo` is ~~injective~~ bijective!

So what?

$\texttt{tree}^7 \simeq \texttt{tree}$?

$|\texttt{tree}^7| = |\texttt{tree}| =$ "countably infinite"

Wait...

```
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  fn (a as N _,b,c,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b as N _,c,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b,c as N _,d,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (a,b,c,d as N _,e,f,g) => N(N(N(N(N(N(g,f),e),d),c),b),a)
   | (E,E,E,E,N(e1,e2),f,g) => N(N(N(N(E,g),f),e1),e2)
   | (E,E,E,E,E,f as N _,g) => N(N(N(N(N(f,g),E),E),E),E)
   | (E,E,E,E,E,E,N(N(N(N(g1,g2),g3),g4),g5)) => N(N(N(N(N(E,g1),g2),g3),g4),g5)
   | (E,E,E,E,E,E,g) => g
```

Wait...

`foo` isn't recursive!

Wait...

$\texttt{tree}^7 \simeq \texttt{tree}$ in constant time!

Recall: Algebraic Datatypes

```
datatype tree = E | N of tree * tree
```

Recall: Algebraic Datatypes

$$T = 1 + T^2$$

$T = \frac{1 \pm \sqrt{-3}}{2}$?

This does give $T^7 = T$...

Does this mean that $T^6 = 1$?

$\texttt{tree}^6 \backsimeq \texttt{unit}$ is obviously false, but it follows from $T = \frac{1 \pm \sqrt{-3}}{2}$

The type equation $T^2 - T + 1 = 0$ is not well-formed, because subtraction isn't defined on types!

What we know

- The only meaningful operations on types are $+$, $\times$ (and exponentiation)
- For any type equation/isomorphism $A = B$, there is a pattern-matching-only bijection from A to B

$$T = T^2 + 1$$

$$T = T^2 + 1 = T^3 + T + 1$$

Attempt 2

$$T = T^3 + T + 1 = T^4 + T^2 + T + 1$$

Attempt 2

(after iterating this a few times...)

$T = T^7 + T^5 + T^4 + T^3 + T^2 + T + 1$

$$T = T^7 + T^5 + T^4 + T^3 + T^2 + T + 1$$

$$T = \color{blue}{T^7} + \color{red}{T^5} + T^4 + \color{red}{T^3} + T^2 + T + 1$$

$$T = {\color{blue}T^7} + {\color{red}T^4} + T^4 + T^2 + T + 1$$

$$T = {\color{blue}T^7} + {\color{red}T^4} + T^4 + {\color{red}T^2} + T + 1 = T^7 + T^4 + {\color{red}T^3} + T + 1$$

$$T = T^7 + T^4 + T$$

Attempt 2

(with the same trick in reverse, you can reduce back to T^7)

- `tree` induces the semiring of polynomials $\mathcal{N}[T]/(T = 1 + T^2)$

Thanks!