Dependent Types

Hype for Types

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Safe Printing

Detypify

Consider these well typed expressions:

```
sprintf "nice"
sprintf "%d" 5
sprintf "%s,%d" "wow" 32
```

What is the type of `sprintf`? Well… it depends.

Types have types too

The type of `sprintf` *depends* on the value of the argument.
In order to compute the type of `sprintf`, we'll need to write a function that takes a string (char list), and returns a *type*!

```
(* sprintf s : formatType s *)

val formatType : char list -> Type = fn
    [] => char list
  | "%"::"d"::cs => int -> formatType cs
  | "%"::"s"::cs => string -> formatType cs
  | _ :: cs      => formatType cs
```

Quantification

Ok, we can express the type of `sprintf s` for some argument `s`, but what's the type of `sprintf`?

Recall that when we wanted to express a type like "`A -> A` for all `A`", we introduced universal quantification over *types*: ∀ `A.A -> A`.

What if we had universal quantification over *values*?

```
sprintf : (s : char list) -> formatType s
```

Curry-Howard Again

What kind of proposition does quantification over values correspond to?

$$(x : \tau) \to A \;\equiv\; \forall x : \tau.A$$

This type can also be written like so:

1. $\forall (x : \tau) \to A$
2. $\forall x : t.A$
3. $\Pi_{x:\tau} A$

Question:

Do we need two kinds of arrow now?
One for dependent quantification and one normal?
Nope!
$A \to B \equiv (\_ : A) \to B$

Some Rules

$$\frac{\Gamma, x : \tau \vdash e : A \quad \Gamma, x : \tau \vdash A : \mathit{Type}}{\Gamma \vdash \lambda(x : \tau)e : (x : \tau) \to A}$$

$$\frac{\Gamma \vdash e_1 : (x : \tau) \to A \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\ e_2 : [e_2/x]A}$$

Vectors Again

If we can write functions from values to types, can we define new types which depend on values?

```
datatype Vec : Type -> Nat -> Type =
  | Nil :  (a : Type) -> Vec a 0
  | Cons : (a : Type) -> (n : Nat) ->
             a -> Vec a n -> Vec a (n+1)

val n = 1 + 2
val xs : Vec string n =
    Cons string 2 "hype" (
        Cons string 1 (Int.toString (n+1)) (
            Cons string 0 "types" (Nil string)))
```

Vectors are actually usable now!

```
val append : (a : Type) -> (n m : Nat) ->
             Vec a n ->
             Vec a m ->
             Vec a (n + m)

val repeat : (a : Type) -> (n : Nat) ->
             a ->
             Vec a n


val filter : (a : Type) -> (n : Nat) ->
             (a -> bool) ->
             Vec a n ->
             Nat × Vec a ??
```

Duality

If we can quantify over the argument to a function, can we quantify over the left element of a tuple?
Yes!

$$(x : \tau) \times A \equiv \exists x : \tau . A$$

This type can also be written:

1. $\{x : \tau \mid A\}$
2. $\Sigma_{x:\tau} A$

As before, $A \times B \equiv (\_ : A) \times B$

```
val filter : (a : Type) -> (n : Nat) ->
             (a -> bool) ->
             Vec a n ->
             (m : Nat) × Vec a m
```

More Rules

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : [e_1/x]A \quad \Gamma, x : \tau \vdash A : Type}{\Gamma \vdash (e_1, e_2) : (x : \tau) \times A}$$

$$\frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_1\ e : \tau} \qquad \frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_2\ e : [\pi_1\ e/x]A}$$

Ok, so what?

Contracts are actually pretty nice

A familiar frustration for 150 students and TAs:

```
(* REQUIRES : input sequence is sorted *)
val search : int -> int seq -> int option

> search 3 [5,4,3] ==> NONE
(* "search is broken!" *)
(* piazza post ensues *)
```

The 122 solution:

```
//@requires is_sorted(xs)
```

Nice, but only works at runtime. What if passing `search` a non-sorted list was *type error*?

A simpler example

```
(* REQUIRES : second argument is greater than zero *)
val div : Nat -> Nat -> Nat
```

Comment contracts are not great, solutions?

```
val div : Nat -> Nat -> Nat option
```

Incurs runtime cost to check for zero, and you still have to fail if it happens.

```
val div : Nat -> (n : Nat) × (1 ≤ n) -> Nat
```

Dividing by zero is impossible! And we incur no runtime cost to prevent it.
What does a value of type $(n : Nat) \times (1 \leq n)$ look like?

$$(3, \text{conceptsHW1.pdf}) : (n : Nat) \times (1 \leq n)$$

Question:

What goes in the PDF?

15-151 Refresher

What constitutes a proof of $n \leq m$?
We just have to define what $(\leq)$ means!

1. $\forall n.\ 0 \leq n$
2. $\forall m\ n.\ n \leq m \Rightarrow n + 1 \leq m + 1$

This looks familiar!

```
datatype (≤) : Nat -> Nat -> Type =
  | LeqZ : (n : Nat) -> 0 ≤ n
  | LeqS : (n : Nat) -> (m : Nat) ->
           n ≤ m -> (n + 1) ≤ (m + 1)
```

$$LeqZ\ 3 : 0 \leq 3$$
$$LeqZ\ 43 : 0 \leq 43$$
$$LeqS\ 0\ 2\ (LeqZ\ 2) : 1 \leq 3$$
$$(3, LeqS\ 0\ 2\ (LeqZ\ 2)) : (n : Nat) \times (1 \leq n)$$

Some Sort of Contract

```
datatype NatList : Type =
  | Nil : NatList
  | Cons : Nat -> NatList -> NatList

datatype Sorted : NatList -> Type =
  | NilSorted : Sorted Nil
  | SingSorted : (n : Nat) -> Sorted (Cons n Nil)
  | ConsSorted : (n m : Nat) -> (xs : NatList) ->
                 n ≤ m ->
                 Sorted (Cons m xs) ->
                 Sorted (Cons n (Cons m xs))

val search : Nat ->
             (xs : NatList) ->
             Sorted xs ->
             Nat option
```

A Type for Term Equality

If we can express a relation like less than or equal, how about equality?

```
datatype Eq : (a : Type) -> a -> a -> Type =
  | Refl : (a : Type) -> (x : a) -> Eq a x x

fun symm (a : Type) (x y : a) :
    Eq a x y -> Eq a y x =
    fn Refl A q => Refl A q

fun trans (a : Type) (x y z : a) :
    Eq a x y -> Eq a y z -> Eq a x z =
    fn Refl A q => fn Refl _ _ => Refl A q


val plus_comm : (n m : Nat) ->
                Eq Nat (n + m) (m + n)
val inf_primes : (n : nat) ->
                 (m : Nat) × ((m > n) × (Prime m))
```