

# Linear Logic and Linear Type Systems

Hype for Types

February 13, 2023

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- A style of logic which treats variables differently than “standard” logic
- How to make `malloc` and `free` safe
- What it looks like to code in a language with resource-aware types

# Linear Logic

# Is Logic Logical?

*The moon is made of green cheese. Therefore, you come to hype for types today.*

## Question

Is this logical?

# Different Interpretation of Implication

Constructive logic interprets  $A \Rightarrow B$  as “If you give me  $A$  is true, then I give you  $B$  is true”.

But what it really says is “If you give me as many copy of  $A$  as I need, then I give you  $B$  is true”.

# Different Interpretation of Implication

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## Idea

The problem of previous example is that to prove the conclusion we only need zero copy of the assumption, hence lacking “relevance”.

## Idea

We need a logic that forces relevance.



# Malloc is Scary...

Consider the following C code:

```
1 int main () {  
2     char *str;  
3     str = (char *) malloc(13);  
4     strcpy(str, "hypefortypes");  
5     free(str);  
6     return(0);  
7 }
```

In C, we have to make sure we allocate and deallocate every memory cell exactly once.

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```

In C, we have to make sure we allocate and deallocate every memory cell exactly once.

## Question

Is there a way to make our *types* guarantee correctness?

# The Problem With Constructive Logic

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## Big Idea

Proofs should no longer be *persistent*, but rather *ephemeral*.

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## Big Idea

Proofs should no longer be *persistent*, but rather *ephemeral*.

Persistence is due to implicit **structural rules**: weakening and contraction.

# Weakening

```
1 int main() {  
2     int *x = (int *) malloc(sizeof(int));  
3     *x = 3;  
4     return 0;  
5 }
```

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```

**Weakening:** we can “drop” assumptions

$$\frac{\Gamma \vdash e : \tau}{\Gamma, x : \tau' \vdash e : \tau} \text{ (WEAK)}$$

# Contraction

```
1 void f(int *x) {  
2     free(x);  
3 }  
4  
5 int main() {  
6     int *x = (int *) malloc(sizeof(int));  
7     *x = 3;  
8     f(x);  
9     f(x);  
10    return 0;  
11 }
```



# Contraction

```
1 void f(int *x) {  
2     free(x);  
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5 int main() {  
6     int *x = (int *) malloc(sizeof(int));  
7     *x = 3;  
8     f(x);  
9     f(x);  
10    return 0;  
11 }
```

**Contraction:** we can “duplicate” assumptions

$$\frac{\Gamma, x_1 : \tau, x_2 : \tau \vdash e : \tau'}{\Gamma, x : \tau \vdash [x, x/x_1, x_2]e : \tau'} \text{ (CNTR)}$$

# Introduction to Linear Logic

In **linear logic**, we have neither weakening nor contraction.

- Requirement that we use each piece of data *exactly* once - no duplication, no dropping
- Comes with an inherent idea of “resources” that are used up
- Allows us to write safe, stateful (imperative!) programs

# The Linear Rules

## Constructive Logic

$$\frac{A \in \Gamma}{\Gamma \vdash A} \text{ (HYP)}$$

# Identity

## Constructive Logic

$$\frac{A \in \Gamma}{\Gamma \vdash A} \text{ (HYP)}$$

## Linear Logic

$$\frac{}{A \vdash A} \text{ (HYP)}$$

# Identity

## Constructive Logic

$$\frac{A \in \Gamma}{\Gamma \vdash A} \text{ (HYP)}$$

## Linear Logic

$$\frac{}{A \vdash A} \text{ (HYP)}$$

### Intuition

“Given  $A$  and nothing else, we can use up  $A$ ”

# Conjunction

## Constructive Logic

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \wedge A_2} (\wedge I)$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_1} (\wedge E1)$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_2} (\wedge E2)$$

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## Linear Logic

$$\frac{\Delta_1 \vdash A_1 \quad \Delta_2 \vdash A_2}{\Delta_1, \Delta_2 \vdash A_1 \otimes A_2} (\otimes I)$$

# Conjunction

## Constructive Logic

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \wedge A_2} (\wedge I)$$

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## Linear Logic

$$\frac{\Delta_1 \vdash A_1 \quad \Delta_2 \vdash A_2}{\Delta_1, \Delta_2 \vdash A_1 \otimes A_2} (\otimes I)$$

$$\frac{\Delta \vdash A_1 \otimes A_2 \quad \Delta', A_1, A_2 \vdash C}{\Delta, \Delta' \vdash C} (\otimes E)$$

# Disjunction

## Constructive Logic

$$\frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \vee A_2} (\vee I_1)$$

$$\frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \vee A_2} (\vee I_2)$$

$$\frac{\Gamma \vdash A_1 \vee A_2 \quad \Gamma, A_1 \vdash B \quad \Gamma, A_2 \vdash B}{\Gamma \vdash B} (\vee E)$$

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## Linear Logic

$$\frac{\Delta \vdash A_1}{\Delta \vdash A_1 \oplus A_2} (\oplus I_1)$$

# Disjunction

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$$\frac{\Delta \vdash A_1 \oplus A_2 \quad \Delta', A_1 \vdash B \quad \Delta', A_2 \vdash B}{\Delta, \Delta' \vdash B} (\oplus E)$$

# Implication

## Constructive Logic

$$\frac{\Gamma, A_1 \vdash A_2}{\Gamma \vdash A_1 \supset A_2} (\supset I)$$

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## Linear Logic

$$\frac{\Delta, A_1 \vdash A_2}{\Delta \vdash A_1 \multimap A_2} (\multimap I)$$

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## Linear Logic

$$\frac{\Delta, A_1 \vdash A_2}{\Delta \vdash A_1 \multimap A_2} (\multimap I)$$

$$\frac{\Delta \vdash A_1 \multimap A_2 \quad \Delta' \vdash A_1}{\Delta, \Delta' \vdash A_2} (\multimap E)$$

# Model Real Worlds Using Linear Logic

5 dollars can buy one coffee and one donut.

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$$\text{\$5} \multimap \text{coffee} \otimes \text{donut}$$

Buffet entrance is 10 dollars. Once you enter, you can eat some beef, and with 2 more dollars you can eat some chicken.

$$\text{\$10} \multimap \text{beef} \otimes (\text{\$2} \multimap \text{chicken})$$

# Towards a Linear $C^0$

---

<sup>0</sup>Fine,  $C^0$ .

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$$\frac{}{\Gamma; x : \tau \vdash x : \tau} \text{ (VAR-LINEAR)}$$

# Resource Splitting: Operators

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$$\frac{\odot : (\tau_1, \tau_2) \rightarrow \tau \quad \Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 \odot e_2 : \tau} \text{ (SML BINOP)}$$

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$$\frac{\odot : (\tau_1, \tau_2) \rightarrow \tau \quad \Gamma; \Delta_1 \vdash e_1 : \tau_1 \quad \Gamma; \Delta_2 \vdash e_2 : \tau_2}{\Gamma; \Delta_1, \Delta_2 \vdash e_1 \odot e_2 : \tau} \text{ (C0 BINOP)}$$

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We also have user-defined top-level functions (e.g. `foo`, `reverse_list`).

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```
1 int* foo(int* a, int* b) {
2   free(a); return b;
3 }
4
5 int main() {
6   int* x = alloc(int);
7   int* y = foo(x, x); // now a type error!
8   free(y);
9   return 0;
10 }
```

## Resource Splitting: Null Checks

In general, pointer equality won't make sense in our language, since all pointers should be distinct.

However, in C, we need a way to check if pointers are NULL! Introducing:

```
1 int* create() /* ... */
2
3 int main() {
4     int* x = create();
5
6     if (x is NULL) {
7         return 0;
8     } else {
9         int y = *x; // still have x here!
10        return y;
11    }
12 }
```

# Resource Splitting: Null Checks

$$\frac{}{\Gamma; ? \vdash \mathbf{NULL} : \tau^*} \text{ (NULL)}$$

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$$\frac{\Gamma; ? \vdash e_1 : \tau_2 \quad \Gamma; ? \vdash e_2 : \tau_2}{\Gamma; \Delta, x : \tau_1^* \vdash \mathbf{ifnull}(x; e_1; e_2)} \text{ (IFNULL)}$$

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# Resource Tracking: Struct Introduction

Just like standard C0, we can allocate structs:

```
1 struct list {  
2     int head;  
3     struct list* tail;  
4 };  
5  
6 struct list* nil() {  
7     return NULL;  
8 }  
9  
10 struct list* cons(int x, struct list* xs) {  
11     struct list* node = alloc(struct list);  
12     node->head = x;  
13     node->tail = xs;  
14     return node;  
15 }
```

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## Problem

We can't eliminate structs like we used to. How will we know that each field is used exactly once?

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We can't eliminate structs like we used to. How will we know that each field is used exactly once?

Structs are just like products - so, pattern match!

```
1 struct list {
2     int head;
3     struct list* tail;
4 };
5
6 int list_sum(struct list* l) {
7     if (l is NULL)
8         return 0;
9
10    let { head = x; tail = xs; } = l; // new syntax
11    return x + list_sum(xs);
12 }
```

# Live Coding

# Conclusion

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- Linear logic is actually all about processes and messages
  - ▶ Concurrency!

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- Linearity as a way of representing state
- Linear propositions in terms of resources
- A practical example of linear logic for memory safety

## Things We Didn't Cover

- Linear logic is actually all about processes and messages
  - ▶ Concurrency!
- Resource tracking (identify the cost of different programs)
- Rust