Dependent Types

Hype for Types

April 4, 2023

Safe Printing

Detypify

Consider these well typed expressions:

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sprintf "nice"
sprintf "%d" 5
sprintf "%s,%d" "wow" 32
```

What is the type of `sprintf`?

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What is the type of `sprintf`? Well... it depends.

Types have types too

The type of `sprintf` *depends* on the value of the argument.
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```
(* sprintf s : formatType s *)

fun formatType (s : char list) : type =
  case s of
    []                  => char list
  | (’%’ :: ’d’ :: cs) => (int -> formatType cs)
  | (’%’ :: ’s’ :: cs) => (string -> formatType cs)
  | (_ :: cs)           => formatType cs
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(* formatType "%d and %s"  =  int -> string -> char list *)
(* sprintf "%d and %s" : int -> string -> char list *)
```

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Ok, we can express the type of `sprintf s` for some argument `s`, but what's the type of `sprintf`?

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What if we had universal quantification over *values*?

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sprintf : (s : char list) -> formatType s
```

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2. $\forall x : t.A$
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Question:

Seems like we now have two arrow types:

1. Normal: $A \to B$.
2. Dependent: $(x : A) \to B$

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Do we need both? Nope!

$$A \to B \equiv (\_ : A) \to B$$

Some Rules

$$\frac{\Gamma, x : \tau \vdash e : A \quad \Gamma, x : \tau \vdash A : \mathit{Type}}{\Gamma \vdash \lambda(x : \tau)e : (x : \tau) \to A} \qquad \frac{\Gamma \vdash e_1 : (x : \tau) \to A \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\ e_2 : [e_2/x]A}$$

Note on Notation

In SML we write type contructors on the *right*:

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val cool : int list = [1,2,3,4]
```

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val cool : List Int = [1,2,3,4]
   val a : A = (* omitted *)
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Question

What is the type of `List`?

```
List : Type -> Type
```

List is a function over types!

Types are values[1]

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Vectors Again

If we can write functions from values to types, can we define new type constructors which depend on *values*?

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```
inductive Vec : Type → Nat → Type
| nil  : (A : Type) → Vec A 0
| cons : (A : Type) → (n : Nat) →
           A → Vec A n → Vec A (n+1)

def xs : Vec String 3 :=
  cons String 2 "hype" (
    cons String 1 (toString 4) (
      cons String 0 "types" (nil String)
    )
  )
```

Vectors Again

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inductive Vec : Type → Nat → Type
| nil  : (A : Type) → Vec A 0
| cons : (A : Type) → (n : Nat) →
            A → Vec A n → Vec A (n+1)

def two := 1 + 0 + 1

def xs : Vec String (6 / two) :=
  cons String two "hype" (
    cons String 1 (toString 4) (
      cons String 0 "types" (nil String)
    )
  )
```

Vectors are actually usable now!

```
val append : (A : Type) -> (n m : Nat) ->
             Vec A n ->
             Vec A m ->
             Vec A (n + m)

val repeat : (A : Type) -> (n : Nat) ->
             A ->
             Vec A n

val filter : (A : Type) -> (n : Nat) ->
             (A -> bool) ->
             Vec A n ->
             Vec A ?? (* What should go here? *)
```

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```

Ponder

How do we describe the return value of `filter`?

Existential Crisis

For filter, we need to return the vector's length, *in addition* to the vector itself:

```
val filter : (A : Type) -> (n : Nat) ->
             (A -> bool) ->
             Vec A n ->
             Nat × Vec A ??
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             Nat × Vec A ??
```

We want to refer to the left value of a tuple, in the TYPE on the right.

Intuition: existential quantification!

There exists some `n : Nat`, such that we return `Vec A n`.

(We're constructivists, so exists means I actually give you the value)

Duality

$$(x : \tau) \times A \equiv \exists x : \tau . A$$

This type can also be written:

1. $\{x : \tau \mid A\}$
2. $\Sigma_{x:\tau} A$

As before, $A \times B \equiv (\_ : A) \times B$

```
val filter : (A : Type) -> (n : Nat) ->
             (A -> bool) ->
             Vec A n ->
             (m : Nat) × Vec A m
```

More Rules

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : [e_1/x]A \quad \Gamma, x : \tau \vdash A : \mathit{Type}}{\Gamma \vdash (e_1, e_2) : (x : \tau) \times A}$$

$$\frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_1 \; e : \tau} \qquad \frac{\Gamma \vdash e : (x : \tau) \times A}{\Gamma \vdash \pi_2 \; e : [\pi_1 \; e/x]A}$$

Ok, so what?

Specifications are actually pretty nice

Discussion

Do you actually read function contracts/specifications in 122/150?

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```
(* REQUIRES : input sequence is sorted *)
val search : int -> int seq -> int option

> search 3 [5,4,3] ==> NONE
(* "search is broken!" *)
(* piazza post ensues *)
```

Compile-time Contracts

The 122 solution:

```
int search (int target, int[] arr)
//@requires is_sorted(arr)
{
    ...
}
```

Nice, but only works at runtime.

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    ...
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```

Nice, but only works at runtime.

What if passing `search` a non-sorted list was a *type error*?

A simpler example

```
(* REQUIRES : second argument is greater than zero *)
val div : Nat -> Nat -> Nat
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```
val div : Nat -> (n : Nat) × (1 ≤ n) -> Nat
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Dividing by zero is impossible! And we incur no runtime cost to prevent it.

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```
val div : Nat -> (n : Nat) × (1 ≤ n) -> Nat
```

Dividing by zero is impossible! And we incur no runtime cost to prevent it.
What does a value of type $(n : Nat) \times (1 \leq n)$ look like?

$$(3, \text{conceptsHW1.pdf}) : (n : Nat) \times (1 \leq n)$$

Question:

What goes in the PDF?

15-151 Refresher

What constitutes a proof of $n \leq m$?

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We just have to define what $(\leq)$ means!

1. $\forall n,\ n \leq n$
2. $\forall m\ n,\ n \leq m \Rightarrow n \leq m + 1$

This looks familiar!

15-151 Refresher

What constitutes a proof of $n \leq m$?
We just have to define what $(\leq)$ means!

1. $\forall n,\ n \leq n$
2. $\forall m\ n,\ n \leq m \Rightarrow n \leq m + 1$

This looks familiar!

```
inductive Le : Nat → Nat → Prop
| refl {n : Nat} : Le n n
| step {n m : Nat} : Le n m → Le n (Nat.succ m)
```

conceptsHW1.pdf

```
inductive Le : Nat → Nat → Prop
| refl {n : Nat} : Le n n
| step {n m : Nat} : Le n m → Le n (Nat.succ m)

def ex1  : Le 0 0 := @Le.refl 0
def ex1' : Le 0 0 := Le.refl

def ex2 : Le 0 3 :=
  Le.step (Le.step (Le.step Le.refl))

def ex3 : Le 1 3 := Le.step (Le.step Le.refl)

def ex4 : (n : Nat) ×' (Le 1 n) :=
  ⟨3, Le.step (Le.step Le.refl)⟩
```

Some Sort of Contract

```
inductive Sorted : List Nat → Prop
| nil_sorted    : Sorted []
| single_sorted : (n : Nat) → Sorted [x]
| cons_sorted   : (n m : Nat) →
                    (xs : List Nat) →
                    Le n m →
                    Sorted (m :: xs) →
                    Sorted (n :: m :: xs)

def search : Nat
          → (xs : List Nat)
          → Sorted xs
          → Option Nat := sorry
```

A Type for Term Equality

If we can express a relation like $\leq$ and sortedness, how about equality?

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```
inductive Eq (A : Type) : A → A → Prop
| refl (a : A) : Eq A a a

def symm (A : Type) (x y : A) (h : Eq A x y) : Eq A y x
  match h with
  | Eq.refl x => Eq.refl

def trans (A : Type) (x y z : A)
          (h1 : Eq A x y) (h2 : Eq A y z)
          : Eq A x z :=
  match h1 with
  | Eq.refl x => h2

def plus_comm : (n m : Nat) → Eq Nat (n + m) (m + n) := sorry

def inf_primes : (n : nat) →
               (m : Nat) ×' ((m > n) × (Prime m)) := sorry
```