Monads

Hype for Types

March 26, 2024

Mappables[1]

[1] Well, "functors", but that's already a thing in SML...

Shmategory Weary

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Suppose we also wanted to "transform" the type `'a` into the type `'a list`.

Question

How would this affect the function `'a -> 'b`? How do we perform the transformation such that the relationship between `'a` and `'b` is preserved?

From Types to Types

Consider the following transformation:

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type 'a map_obj    = 'a list
fun     map_arr f = List.map f
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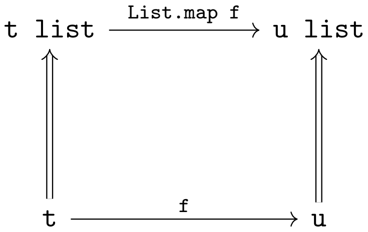
```

where we

- take a type `t` and turn it into type `t list`
- take a function `f : t -> u` and turn it into a function `List.map f : t list -> u list`

Visualizing Lists



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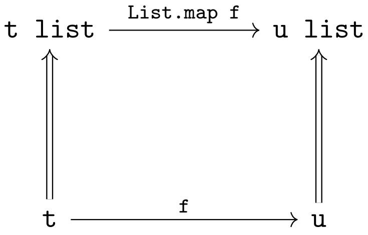

Key Idea

Even though the types 'a and 'b are now different, the relationship between them has been preserved by the transformation.

Mappables?

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- type `'a t`
- value `map : ('a -> 'b) -> 'a t -> 'b t`

In other words:

```
signature MAPPABLE =
  sig
    type 'a t
    val map : ('a -> 'b) -> 'a t -> 'b t
  end
```

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```
type 'a map_obj   = 'a list

fun map_arr1 f =
  fn _ => []
fun map_arr2 f =
  fn l => List.map f (List.rev l)
fun map_arr3 f =
  fn []     => []
   | _::xs => List.map f xs
```

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- The identity function `id : 'a -> 'a` is transformed into the identity function `id' : 'a t -> 'a t` for the new type

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We would want our transformation on functions to maintain the following:

- The identity function `id : 'a -> 'a` is transformed into the identity function `id' : 'a t -> 'a t` for the new type
- For any functions `f : 'a -> 'b` and `g : 'b -> 'c`, `map (f o g) = (map f) o (map g)`

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In other words:

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signature MAPPABLE =
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    type 'a t
    val map : ('a -> 'b) -> 'a t -> 'b t
    (* invariants: ... *)
  end
```

Optimization: Loop Fusion!

If we have:

```
int[n] arr;

for (int i = 0; i < n; i++)
  arr[i] = f(arr[i]);

for (int i = 0; i < n; i++)
  arr[i] = g(arr[i]);
```

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then it must be equivalent to:[2]

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Some Example Mappables

- Lists
- Options
- Trees
- Streams
- Functions `int -> 'a`
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i.e., (almost) anything polymorphic.

Conclusion

It's a useful abstraction!

Monads

Descent into partial madness

Partial functions return options

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- `tail : 'a list -> 'a list option`

How would we write the partial version of `tail_3`?

```
(* tail_3 : 'a list -> 'a list *)
fun tail_3 (_::_::_::L) = L
```

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  | SOME L1 =>
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What about tail_5?

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Let's consider another kind of compose:

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o :  ('b -> 'c) * ('a -> 'b) -> 'a -> 'c
<=< :  ('b -> 'c opt) * ('a -> 'b opt) -> 'a -> 'c opt
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o :  ('b -> 'c) * ('a -> 'b) -> 'a -> 'c
<=< :  ('b -> 'c opt) * ('a -> 'b opt) -> 'a -> 'c opt
```

Ta-da!

```
fun f <=< g =
  (fn NONE => NONE | SOME x => f x) o g
```

More than a composition

Some useful versions of common tools

```
type 'a t = 'a option
```

Compose

```
val <=< :
  ('b -> 'c t) * ('a -> 'b t) -> ('a -> 'c t)
```

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val <=< :
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Apply

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val >>= : 'a t * ('a -> 'b t) -> 'b t
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val <=< :
  ('b -> 'c t) * ('a -> 'b t) -> ('a -> 'c t)
```

Apply

```
val >>= : 'a t * ('a -> 'b t) -> 'b t
```

Flatten

```
val join : 'a t t -> 'a t
```

bind : 'a t * ('a -> 'b t) -> 'b t

```
type 'a t = 'a option
fun x >>= f = case x of SOME x => f x
                      | NONE => NONE
```

```
bind : 'a t * ('a -> 'b t) -> 'b t
```

```
type 'a t = 'a option
fun x >>= f = case x of SOME x => f x
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type 'a t = 'a list
fun xs >>= f = List.concat (List.map f xs)
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type 'a t = 'a * string
fun (x,a) >>= f = let (y,b) = f x
                  in (y,a^b) end
```

bind : 'a t * ('a -> 'b t) -> 'b t

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type 'a t = 'a * string
fun (x,a) >>= f = let (y,b) = f x
                  in (y,a^b) end
```

```
type 'a t = unit -> 'a
fun x >>= f = fn () => f (x()) ()
```

bind : 'a t * ('a -> 'b t) -> 'b t

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type 'a t = 'a list
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type 'a t = unit -> 'a
fun x >>= f = fn () => f (x()) ()

datatype 'a t = Ret of 'a | Err of exn
fun x >>= f = case x of Ret a => f x
                      | Err x => Err x
```

Example: Errors

```
type 'a t = 'a option
fun x >>= f = case x of SOME x => f x
                      | NONE => NONE
fun bind (x, f) = x >>= f

fun divide(x : int, y : int) : int t =
  if y = 0 then NONE
  else SOME (x div y)

val _: string t =
  bind (divide (10, 3), fn x =>
  bind (Int.fromString "0", fn y =>
  bind (divide (x, y), fn z =>
  SOME (Int.toString (x + y + z)))))
```

Example: Printing

```
type 'a t = string * 'a
fun bind (e : 'a t, f : 'a -> 'b t) : 'b t =
  let
    val (s1, a) = e
    val (s2, b) = f a
  in
    (s1 ^ s2, b)
  end
```

Example: Printing

```
fun print (s : string) : unit t =
  (s, ())
fun add (x : int, y : int) : int t =
  ("adding", x + y)
val _ : int t =
  bind (print "hi", fn () =>
  bind (add (20, 22), fn n =>
  ("done", n)))
(* result : ("hiaddingdone", 42) : int t *)
```

Programming with Monads

```
readInput      : stream -> string option
parseUsername  : string -> string option
getUserFromId  : string -> user option
getAvatar      : user   -> image option
```

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readInput      : stream -> string option
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SOME TextIO.stdIn
  >>= readInput
  >>= parseUsername
  >>= getUserFromId
  >>= getAvatar
```

Parallel: Imperative Programming

```
inString <- SOME TextIO.stdIn
userId <- parseUsername inString
user <- getUserFromId userId
avatar <- getAvatar user
```

Useful pattern!

Key Idea

Monads are a useful programming tool!

```
signature MONAD =
  sig
    type 'a t
    val return : 'a -> 'a t
    val bind : 'a t * ('a -> 'b t) -> 'b t
  end
```

Monads are like burritos

A monad is a special kind of a functor. A functor F takes each type T and maps it to a new type FT. A burrito is like a functor: it takes a type, like meat or beans, and turns it into a new type, like beef burrito or bean burrito.

Monads are like burritos

A functor must also be equipped with a **map** *function that lifts functions over the original type into functions over the new type. For example, you can add chopped jalapeños or shredded cheese to any type, like meat or beans; the lifted version of this function adds chopped jalapeños or shredded cheese to the corresponding burrito.*

Monads are like burritos

A monad must also possess a **return** *function that takes a regular value, such as a particular batch of meat, and turns it into a burrito. The unit function for burritos is obviously a tortilla.*

Monads are like burritos

Finally, a monad must have a **bind** *function that takes a burrito, tells you how to shuffle the ingredients, and turns it into a new burrito. For example, given a burrito, you can unwrap the tortilla, add cheese, and rewrap it.*

Monads are like burritos

The **map**, **bind**, *and* **return** *functions must satisfy certain laws. For example, if* **B** *is already a burrito, and not merely a filling for a burrito, then* **B** *>>=* **return** *must be the same as* **B**. *This means that if you have a burrito, unwrap the burrito, and rewrap it in a new tortilla, its the same as the original burrito.*