Automatic Theorem Provers 2 + Dependent Types

Hype for Types

March 9, 2026

Review: Last Week

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UNSAT is verification

Ask: "is there any input where property P fails?"

UNSAT $\Rightarrow$ P holds for *all* inputs. Automatically.

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The result

NP-complete in theory. Millions of variables in seconds in practice.

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Z3 (Microsoft Research)

The dominant SMT solver. Handles integers, arrays, bitvectors, strings.

Ask: "is there any input where this property fails?"

`unsat` = proved correct for all inputs. `sat` = here's your counterexample.

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The simplest example

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-- The TYPE is the claim: for any Nat n, n + 0 equals n
+-zero : (n : Nat) -> n + 0 == n
+-zero zero     = refl                          -- 0 + 0 = 0, by
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+-zero (suc n) = cong suc (+-zero n)    -- inductive step
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The key difference

Tests check $n = 0, 1, 2, \ldots$ and hope. This proves it for every n. No exceptions.

The +-comm Proof: Why It's Non-Trivial

Why does $n + m = m + n$ need work? Look at how + is defined:

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zero  + m = m                 -- 0 + m just returns m
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+-comm : (n m : Nat) -> n + m == m + n
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    lemma
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The recursive call `+-comm n m` *is* the inductive hypothesis. Induction is just recursion.

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Proof	Program	body of `+-comm`
True	Inhabited type	`Nat`, `Bool`
$P \Rightarrow Q$	`P -> Q`	`sym`, `trans`
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Every row is a function you've already written. The table just tells you what it means logically.

What Regular Types Can and Cannot Say

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The thing types check: kind

Simple types answer "what kind of value?" — integer, string, list, function.
They are completely silent about *which* value, or what's true about it.

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The pattern

Every one of these is a condition on a *value*, not on its kind.
The type "list of integers" says nothing about length, order, or membership.
You need a type that can reference the actual value.

Where There Is Room to Change

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Today: structure only

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- `List[int]`: any such list
- `int -> int`: any such function

The value is irrelevant to the type.
Types and values live in separate worlds.

What we could express

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- a list *of length exactly n*
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The gap that dependent types fill

In a simple type system, there is no slot in the type language to write "length = 5" or "output is sorted." Dependent types add that slot. The type can now mention a specific value.

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Type checking must terminate.
A type whose membership requires solving the halting problem is not allowed.
Agda's termination checker enforces this.

Fixed: checked at compile time

If checking required running the program, you'd lose the whole benefit.

Fixed: soundness

The type system must never lie.
If it certifies a value has type T, it must actually have type T.

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The dependent types move in one sentence

Allow the type language to reference values from the value language.
Keep termination, compile-time checking, and soundness intact.

Dependent Types

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`Vec Int 5` says: a list of *exactly five* integers. The type carries a guarantee.
The type checker can now enforce facts about values, not just kinds.

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datatype zero  = ZERO_  (* never constructed *)
datatype 'n succ = SUCC_ (* never constructed *)

(* 'a vlist carries a phantom 'n tracking length *)
datatype ('a, 'n) vlist =
    Nil  : ('a, zero) vlist
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The length tag lives entirely in the type. At runtime, `zero` and `succ` vanish.

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    Nil  : ('a, zero) vlist
  | Cons : 'a * ('a, 'n) vlist -> ('a, 'n succ) vlist
```

`Nil` has type `('a, zero) vlist`.
`Cons(1, Nil)` has type `(int, zero succ) vlist`.
The length tag lives entirely in the type. At runtime, `zero` and `succ` vanish.

Now `head` can demand non-empty

```
fun head (Cons(x, _) : ('a, 'n succ) vlist) = x
```

Passing `Nil` is a type error — `zero` $\neq$ `'n succ`.
This *looks* like a dependent type. It isn't. We'll see why.

Phantom Types: Where the Illusion Breaks

The problem: the tag is decoration. SML can't compute with it.

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append — what type does it return?

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(* We want: ('a, 'm) vlist -> ('a, 'n) vlist
                              -> ('a, 'm + 'n) vlist *)
(* But 'm + 'n is not a type expression in SML. *)
(* The best we can write: *)
fun append Nil          ys = ys
  | append (Cons(x,xs)) ys = Cons(x, append xs ys)
(* Return type? SML infers ('a, 'n succ) vlist ---
   it loses the arithmetic entirely.                *)
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The fundamental wall

Phantom types can enforce *fixed* structural constraints (non-empty, same tag).

They cannot express *computed* relationships like $m + n$, $n - 1$, or $n < k$.

The tags are inert. You can't do arithmetic on types in SML.

The Gap: Types and Values Are Separate Worlds

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SML: two separate universes

Types live at compile time.
Values live at runtime.

`zero` as a *type*: a compile-time label.
`0` as a *value*: a runtime integer.

They are completely unrelated.
You cannot use a value inside a type.
You cannot compute a type from a value.

Agda: one universe

`Nat` is a type *and* its values are usable inside other types.

`Vec A 3`: the 3 here is a genuine `Nat` value sitting inside a type.

The type checker can evaluate `m + n`.
It knows `Vec A (2 + 3) = Vec A 5`.
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The dependent types move — precisely stated

Collapse the wall between the type universe and the value universe.
Let types be indexed by *actual values*. Then `Vec A n` means what it says.

So What's the Actual Difference?

Both phantom types and dependent types let you write `head` safely. The difference shows up the moment you try to *compute* with the index.

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Phantom type — a label

`'n` is a compile-time tag.
`zero succ` means "tagged once."
It is *not* the number 1.

The type checker can match tags.
It **cannot** add them.

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(* can't write this in SML *)
'm + 'n
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`'m + 'n` is not a type expression.

Dependent type — a value

`n` is a real `Nat`.
3 means the number 3.
It *is* the value.

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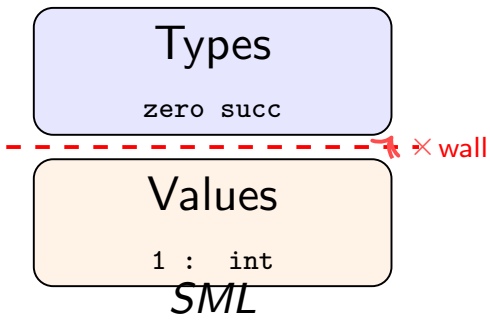

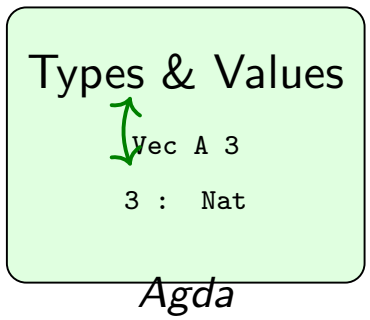

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Vec A n: a list of *exactly* n elements, where n is baked into the type itself.

```
data Vec (A : Set) : Nat -> Set where
  []    : Vec A zero                  -- empty list has length
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  _::_  : A -> Vec A n -> Vec A (suc n)   -- cons adds
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head : Vec A (suc n) -> A    -- only callable on length >=
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Now head is total

```
head : Vec A (suc n) -> A    -- only callable on length >= 1
```

`Vec A zero` does not unify with `Vec A (suc n)`.
Passing an empty vector is a **compile-time type error.**
Not a Maybe. Not a crash. The bad call is *inexpressible*.

Vec: Relationships Between Lengths

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Return type is `m + n`. The type checker verifies the arithmetic.
A buggy append that silently drops elements would fail to typecheck.

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Fin n: the type of natural numbers strictly less than n.

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An out-of-bounds index **has no type**. It cannot be written.

The Limits

The Specification Problem

There's a limit that no amount of formal machinery fixes. Here it is.

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What you actually get from formal verification

A proof that your code matches your spec.
Not that your spec matches the real world.

Writing the spec correctly is the hard part. The proof doesn't help with that.

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System	What it can express	Checked when
No types	Nothing	Never
Simple types	What *kind* of value	Compile time
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Next time

Types that describe not *what* a value is, but *how* it gets used.
Effect types, linear types, session types — same idea, different axis.