

Constructive Logic

*-317

February 3, 2025

Proofs

Existence

I want to prove there exists a set with property P .

Is one of these more useful?

- Proof by contradiction: If such a set did not exist, we'd have a contradiction (insert proof here), therefore it must exist
- Direct proof: The set S has property P (insert proof here)

Existence

I want to prove there exists an algorithm to convert SML into x86 assembly.

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Existence

I want to prove there exists an algorithm to convert SML into x86 assembly.

- Proof by contradiction: If such a compiler did not exist, we'd have a contradiction (insert proof here), therefore it must exist
- Direct proof : CakeML (formally verified SML compiler)

To Construct or Not To Construct

Two kinds of proofs

- *Non-Constructive* : demonstrate the existence of a mathematical object, but *without* telling you what it is
- *Constructive* : demonstrate the existence of a mathematical object precisely by presenting an object and proving it has the desired properties

A Rational Example

Does there exist $a, b : \mathbb{R}$ such that a, b irrational but a^b rational?

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- *Non-Constructive*

If $\sqrt{2}^{\sqrt{2}}$ rational, we're done. Otherwise $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$.

A Rational Example

Does there exist $a, b : \mathbb{R}$ such that a, b irrational but a^b rational?

- *Non-Constructive*

If $\sqrt{2}^{\sqrt{2}}$ rational, we're done. Otherwise $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$.

- *Constructive*

Take $a = \sqrt{2}$ and $b = \log_2 9$. Then $\sqrt{2}^{\log_2 9} = 9^{\log_2 \sqrt{2}} = 9^{\frac{1}{2}} = 3$

Non-constructive proofs are frustrating in the real world

- Brouwer's Fixed Point Theorem
- Expander Graphs
- Ramsey Theory

Constructive proofs are useful to computer scientists

Constructive proofs provide *algorithms!*

A proof that all natural numbers have property P must describe a way to construct a proof of $P(n)$ for each $n : \mathbb{N}$

Formalization B)

- “You have to construct something” is pretty vague
- How do we formalize what it means for a proof to be constructive?

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- “You have to construct something” is pretty vague
- How do we formalize what it means for a proof to be constructive?
 - 1 Decide what kinds of proposition we want to talk about
 - 2 Inference rules!

Formalization B)

A few reasonable kinds of proposition

- \top
- \perp
- $A \wedge B$
- $A \vee B$
- $A \supset B$
- $\neg A$

Constructive Logic: Inference Rules

Conjunction (\wedge)

To get $A \wedge B$ (*introduction*), we need...

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To get $A \wedge B$ (*introduction*), we need... a proof of A and a proof of B :

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

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Given $A \wedge B$, we can extract two facts (*elimination*)...

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Given $A \wedge B$, we can extract two facts (*elimination*)... A and B :

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} (\wedge E_1)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} (\wedge E_2)$$

Implication (\supset)

To get $A \supset B$, we need...

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To get $A \supset B$, we need... a proof of B *assuming* a proof of A :

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To get $A \supset B$, we need... a proof of B *assuming* a proof of A :

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Implication (\supset)

To get $A \supset B$, we need... a proof of B *assuming* a proof of A :

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} (\supset I)$$

Given $A \supset B$ and A , we can extract... B :

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} (\supset E)$$

Disjunction (\vee)

To get $A \vee B$, we need...

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To get $A \vee B$, we need... a proof of A or a proof of B :

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Given $A \vee B$, we can extract... nothing?

Disjunction (\vee)

To get $A \vee B$, we need... a proof of A or a proof of B :

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Given $A \vee B$, we can extract... nothing? But if we *also* have “given A , then C ” and “given B , then C ,” we can get C :

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} (\vee E)$$

Truth (\top) and Falsehood (\perp)

To get \top , we need...

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

$$\frac{}{\Gamma \vdash \top} (\top I)$$

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

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Can we get any information out of \top ?

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

$$\frac{}{\Gamma \vdash \top} (\top I)$$

Can we get any information out of \top ? No!

How can we get \perp ?

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

$$\frac{}{\Gamma \vdash \top} (\top I)$$

Can we get any information out of \top ? No!

How can we get \perp ? We can't!

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

$$\frac{}{\Gamma \vdash \top} (\top I)$$

Can we get any information out of \top ? No!

How can we get \perp ? We can't!

But given \perp , we can obtain a proof of...

Truth (\top) and Falsehood (\perp)

To get \top , we need... nothing!

$$\frac{}{\Gamma \vdash \top} (\top I)$$

Can we get any information out of \top ? No!

How can we get \perp ? We can't!

But given \perp , we can obtain a proof of... anything!

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash A} (\perp E)$$

What about Negation (\neg)?

What counts as a proof of $\neg A$?

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What counts as a proof of $\neg A$?

We need to show something like "it's impossible to prove A "

Do we need new inference rules?

No!

$$\neg A \equiv A \supset \perp$$

$\neg A$ means "If we can prove A , we can do something impossible".

All the rules!

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

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Question

Does this seem familiar...?

Programs are Proofs

Back to the Simply-Typed Lambda Calculus

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Back to the Simply-Typed Lambda Calculus

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Let's Prove Some Stuff!

Some Examples

Theorem: Identity

Prove $A \supset A$.

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$\lambda f : A \times B \rightarrow C. \lambda a : A. \lambda b : B. f \langle a, b \rangle$, or `Fn.curry` in SML

More Examples

Theorem

Prove $\perp \vee T$.

More Examples

Theorem

Prove $\perp \vee T$.

Right $\langle \rangle$

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Prove $\perp \vee \top$.

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Prove $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$.

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$\lambda x : A \times (B + C)$. **case** $\text{snd}(x)$ **of** $x_1 \Rightarrow$ **Left** $\langle \text{fst}(x), x_1 \rangle \mid x_2 \Rightarrow$

Right $\langle \text{fst}(x), x_2 \rangle$

$\lambda x : (A \times B) + (A \times C)$. **case** x **of** $x_1 \Rightarrow \langle \text{fst}(x_1), \text{Left } \text{snd}(x_1) \rangle \mid x_2 \Rightarrow$
 $\langle \text{fst}(x_2), \text{Right } \text{snd}(x_2) \rangle$

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Is there anything we can't prove constructively?

- Law of Excluded Middle : $P \vee \neg P$
- Double Negation Elimination : $\neg\neg P \supset P$

(These are actually equivalent)

So what?

Practical Implications



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Question

These proofs seem pretty boring, can a type system express more complicated propositions?