Polymorphism: What's the deal with 'a?

Hype for Types

March 10, 2025

Polymorphism

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Recall lambda abstraction from the Simply Typed Lambda Calculus

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If we want it to work for Bools, we'd have to write a separate function:

$$id2 = \lambda(x : \texttt{Bool})x$$

This seems really annoying $>$: (

What does SML do?

```
val id = fn (x : 'a) => x
val _ = id 1
val _ = id true
val _ = id "nice"
```

```
id :  'a -> 'a
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But what *is* `'a`? Is it a type?

If `id 1` type checks then `1 : 'a`???

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The ticks are no longer needed, as we've explicitly bound a as a type variable.

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$$(\Lambda(a : \text{Type})\lambda(x : a)x)[\texttt{Nat}] \Longrightarrow \lambda(x : \texttt{Nat})x$$

System F

The polymorphic lambda calculus we've developed is called System F.
Let's write a grammar!

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$$
\begin{array}{rcll}
e & ::= & x & \text{term variable} \\
 & | & \lambda(x : \tau)e & \text{term abstraction} \\
 & | & \Lambda(t : \text{Type})e & \text{type abstraction} \\
 & | & e_1\, e_2 & \text{term application} \\
 & | & e_1[\tau] & \text{type application}
\end{array}
$$

$$
\begin{array}{rcll}
\tau & ::= & t & \text{type variable} \\
 & | & \tau_1 \to \tau_2 & \text{function type} \\
 & | & \forall t.\tau & \text{polymorphic type}
\end{array}
$$

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Do we need anything else? What about product types? Sum types?

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We'll get back to that later...

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$$\Lambda(a\ b\ c : \text{Type})\lambda(f : a \to b)\lambda(g : b \to c)\lambda(x : a)g(f\ \ x)$$

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Consider:

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Type error! In SML, big lambdas can only be present at *declarations*, not arbitrarily inside expressions. Our function here is equivalent to:

$$hmm = \Lambda(a : \text{Type})\lambda(id : a \rightarrow a)(id\ 1, id\ \ true)$$

Which is *not* the same as:

$$hmm = \lambda(id : \forall a.a \rightarrow a)(id[int]\ 1, id[bool]\ \ true)$$

Why? Because type inference for System F is undecidable!

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$\forall t.t \rightarrow t$ means “for *any* type `t`: if you give me a `t`, I’ll give you a `t`”

So $\exists t.t \rightarrow t$ should probably mean “there is some *specific* type `t`, and if you give me that `t`, I’ll give you a `t`”

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So $\exists t.t \to t$ should probably mean "there is some *specific* type `t`, and if you give me that `t`, I'll give you a `t`"

Question

Does this sound similar to anything in SML?

Existentialism == Modules!

```
signature S =
  sig
    type t
    val x : t
    val f : t -> t
  end
```

is basically equivalent to:

$$\exists t.\{x : t, f : t \to t\}$$

or even more simply:

$$\exists t.t \times (t \to t)$$

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Main Idea

We use **signatures** to represent **existential types**!

Existentialism == Modules!

Question

What is a value of type $\exists t.\tau$?

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Answer: A module!

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```
structure M : S =
  struct
    type t = int
    val x = 150
    val f = fn x => x + 1
  end
```

is a value of type $\exists t.\{x : t, f : t \to t\}$

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In other words, I obtain the type `t` and value of type `t * (t -> t)` that `M` implements!

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Main Idea

opening a value (module) of type $\exists t.\tau$ gives us a type t and a value of type τ

Typechecking Rules

$$\frac{\Delta, t \vdash \tau \ \ type}{\Delta \vdash \exists t.\tau \ \ type} \qquad \frac{\Delta; \Gamma \vdash e : [\rho/t]\tau \quad \Delta \vdash \rho \ \ type}{\Delta; \Gamma \vdash struct \ \ type \ \ t = \rho \ \ in \ \ e : \exists t.\tau}$$

$$\frac{\Delta; \Gamma \vdash M : \exists t.\tau \quad \Delta, t; \Gamma, x : \tau \vdash e : \tau' \quad \Delta \vdash \tau' \ \ type}{\Delta; \Gamma \vdash open \ \ M \ \ as \ \ t, x \ \ in \ \ e : \tau'}$$

Example: Stacks!

```
signature STACK =
  sig
    type t
    val empty : t
    val push : int -> t -> t
    val pop : t -> (int * t) option
  end

structure ListStack : STACK =
  struct
    type t = int list
    val empty = []
    fun push x xs = x :: xs
    fun pop [] = NONE
      | pop (x :: xs) = SOME  (x, xs)
  end
```

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$$struct\ \ type\ \ t = int\ \ list\ \ in$$

$$\{empty = Nil,$$

$$push = Cons,$$

$$pop = ...\}$$

What about functors?

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signature STACK =
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    type t
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  end

functor MkDoubleStack (S : STACK) : STACK =
  struct
    type t = S.t
    val empty = S.empty
    fun push x s = S.push x (S.push x s)
    val pop = S.pop
  end
```

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$$open\ S\ as\ t', s\ in$$

$$struct\ type\ t = t'\ in$$

$$\{empty = s.empty,$$

$$push = \lambda(x : int).(s.push\ x)\ o\ (s.push\ x)$$

$$pop = s.pop\}$$

We don't need no type constructors (except $\forall$ and $\rightarrow$)

Question

Can we encode $A \times B$ in System F?

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Idea

A product is defined by the fact that, given a value of type $A \times B$, we have access to both a value of type A and a value of type B

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Product Types in System F

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$$pair : \forall A\ B.A \to B \to A \times B =$$

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$$\Lambda(A\ B)\ \lambda(p : A \times B)\ p[A]\ (\lambda(x : A)\ \lambda(y : B)\ x)$$

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Answer: An encoded value of type $A + B$ *is already* a case!